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Regularization of non-commutative SYM by orbifolds with discrete torsion and $\mathrm{SL}(2,\mathbb{Z})$ duality

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Abstract: We construct a nonperturbative regularization for Euclidean noncommutative supersymmetric Yang-Mills theories with four ($\mathcal{N} = (2,2)$) , eight ($\mathcal{N} = (4,4)$) and sixteen ($\mathcal{N} = (8,8)$) supercharges in two dimensions. The construction relies on orbifolds with discrete torsion, which allows noncommuting space dimensions to be generated dynamically from zero dimensional matrix model in the deconstruction limit. We also nonperturbatively prove that the twisted topological sectors of ordinary supersymmetric Yang-Mills theory are equivalent to a noncommutative field theory on the topologically trivial sector with reduced rank and quantized noncommutativity parameter. The key point of the proof is to reinterpret 't Hooft's twisted boundary condition as an orbifold with discrete torsion by lifting the lattice theory to a zero dimensional matrix theory.

Keywords: Extended Supersymmetry, Duality in Gauge Field Theories, Non-Commutative Geometry, Lattice Gauge Field Theories.

Contents

1. Introduction

Supersymmetric Yang-Mills theories on noncommutative space-time geometries appeared in string theory via the toroidal compactification of Matrix theory [1, 2] with a certain background flux, as well as through quantization of open strings ending on D branes in the presence of B-fields along the world volume of D brane [4, 3]. From a field theoretic perspective, field theories on a noncommutative space [5, 6] are interesting as a new universality class. Even though these theories are nonlocal and lack Lorentz invariance, they are believed to be sensible quantum field theories because of their UV completion. (See reviews [7, 9, 8].)

One of the main problems to understand is the existence of the quantized noncommutative field theory. To answer this question requires a nonperturbative definition. The renormalization group properties of noncommutative field theories may be used with a nonperturbative regulator to address the existence problem. Proper understanding of the renormalization group flow for these theories is also a fundamental issue. As noncommutative theories show interesting UV and IR mixing, their renormalization group properties might in principle be quite different from local quantum field theories [11, 10].

Solid progress have been made by the proposal of a constructive definition of noncommutative pure Yang-Mills theory [13 – 16]. Remarkable properties of these theories such as Morita equivalence (T duality), UV/IR mixing characteristic have been shown nonperturbatively [14, 15]. As a special case, it has been shown that the twisted Eguchi-Kawai model [18, 17] is Morita equivalent to a noncommutative lattice gauge theory [14, 16].

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However, the noncommutative gauge theories emerging as limits of string theory are generally extended supersymmetric Yang-Mills theories. For example, the SYM theory on a cluster of D branes is the maximally supersymmetric gauge theory with sixteen supersymmetries. Even without incorporating noncommutativity, a nonperturbative definition of supersymmetric theories is a long standing problem. Recently, a lattice regularization of ordinary SYM theories was presented for both spatial [19] and Euclidean lattices [20 – 22]. The supersymmetric lattices are obtained by an orbifold projection of a supersymmetric parent matrix theory that has as much supersymmetry as the target theory. The projection leaves a subset of the supersymmetries (as few as one exact supersymmetry) realized exactly on lattice, protecting the theory from unwanted relevant operators in the continuum limit. The orbifold projection technique had been introduced in the field theory language in [27] and the supersymmetric lattice constructions are motivated by deconstruction of supersymmetric gauge theories [28, 29].[1] The construction for supersymmetric spatial lattices had been extended to incorporate noncommutative spatial lattices in [30]. In particular, they construct $\mathcal{N} = 8$ noncommutative SYM theory in $d = 3$ by generalizing the results in [19]. Also see [31, 32].

The primary purpose of this work is to construct a full non-perturbative regulator for noncommutative euclidean extended SYM theories $\mathcal{N} = (2, 2)$, $\mathcal{N} = (4, 4)$ and $\mathcal{N} = (8, 8)$ in $d = 2$ dimensions with gauge group $\mathrm{U}(k)$. These theories are the dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theories from $d = 4, 6, 10$ dimensions respectively to $d = 2$ dimensions. Since we are concerned about Euclidean theories, in principle, we evade the questions associated with timelike noncommutativity [33, 34] henceforth it will not be discussed here.

We use orbifold projection with discrete torsion [35 – 39] to generate the regulated noncommutative SYM theory. But a naive implementation generates a noncommutative moduli space which does not lead to a sensible continuum limit, i.e. the deconstruction limit does not generate the target theory with a noncommutative base space. In order to obtain a sensible continuum, it is necessary to start with a variant of the parent theory, which can be thought of as a deformation of the "superpotential" (in the context of spatial noncommutative supersymmetric lattices, this has been suggested in [30].) The key issue here is that orbifold theory is a priori a matrix theory which has the topology of a discrete torus but no dimensionful parameter to be identified as a lattice spacing. The vacuum expectation value of the moduli fields determine the inverse lattice spacing under appropriate circumstances, hence far from the origin of moduli space corresponds to taking the lattice spacing to zero. Without the deformation, one of the moduli fields is frozen at the origin and hence there is no lattice interpretation.

The deformed parent matrix theory has much less supersymmetry than its standard counterpart. But orbifold projection is arranged in such a way that the daughter theory respects the same supersymmetry of the deformed theory. In other words, if the parent

[1] An alternative approach to lattice supersymmetry based on twisted supersymmetry algebra is given in [23 – 26] and references therein.

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theory were not deformed, the orbifold projection would break all the supersymmetries but the ones which are compatible with deformed parent theory. The standard parent matrix theories possess $\mathcal{Q} = 4, 8, 16$ supersymmetries, whereas their deformed counterparts have $\mathcal{Q} = 1, 2, 4$ exact supersymmetries which is also the amount of supersymmetry inherited by the resulting $d = 2$ dimensional lattices. These lattices support the $\mathcal{N} = (2,2)$, $\mathcal{N} = (4,4)$ and $\mathcal{N} = (8,8)$ noncommutative SYM theories, respectively in their moduli space.

The orbifold projection with discrete torsion of the deformed parent matrix theory is the lattice regularization of the noncommutative SYM theory on a discrete torus. The complete set of solutions of orbifold constraints form the basis for the noncommutative lattice. The algebra of (matrix valued) functions on the noncommutative discrete torus is in fact the same as the algebra of ordinary (matrix valued) functions on the commutative discrete torus where the usual (matrix) product is modified to the lattice Moyal $\star$ product. This construction will basically form a full nonperturbative regulator for Euclidean noncommutative SYM theories in two dimensions.

The second main objective of our work is to show that the nontrivial topological sectors of ordinary extended SYM theories are equivalent to noncommutative SYM theories with reduced rank at a nonperturbative level. To achieve this goal, we consider ordinary extended SYM theories with background nonabelian magnetic fluxes, which correspond to the nontrivial topological sectors of the theory [40, 41]. Since the matter fields (both fermions and bosons) in all of our target theories are in adjoint representation, and hence are invariant under the center of the gauge group, we have a $(\mathrm{U}(1) \times \mathrm{SU}(k))/\mathbb{Z}_k$ theory. This in turn allows us to choose different classes of boundary conditions for the superfields, without harming the exact supersymmetry [42] of the lattice. By solving twisted boundary conditions, we nonperturbatively prove that an ordinary rank k extended SYM theory with a definite topological charge q is equivalent to a *noncommutative* SYM theory with a reduced rank k_0 and periodic boundary conditions on a larger size torus. The pivotal point of the proof is realizing that *'t Hooft's twisted boundary condition is in fact an orbifold with discrete torsion when the lattice gauge theory is expressed as a zero dimensional matrix theory.* More interestingly, the discrete torsion (under the given circumstances) turns out to be topological and is given in terms of the topological numbers of the ordinary SYM theory as the ratio of integer flux q to the rank k of gauge group or equivalently, the ratio of first Chern character to zeroth Chern character.

The organization of the paper is as follows: in section 2, the technique for obtaining a noncommutative lattice from orbifold with discrete torsion is reviewed. In section 3, we construct a lattice action for the noncommutative $\mathcal{N} = (2,2)$ theory in $d = 2$ dimensions in detail. We show that the continuum limit with finite volume is compatible with a finite noncommutativity parameter, and this yields the $\mathcal{N} = (2,2)$ theory in two dimensions. In section 4, we prove the $\mathrm{SL}(2,\mathbb{Z})$ duality (which is part of T duality) nonperturbatively. In section 5 and 6, we give the results for noncommutative $\mathcal{N} = (4,4)$ and $\mathcal{N} = (8,8)$ theories without too much detail. Then, we discuss some implications of results.

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2. Non-commutative lattice from orbifold with discrete torsion

In this section, we first solve the orbifold conditions and obtain a complete and orthonormal basis for the noncommutative two dimensional lattice. Generalization to higher dimensions is straightforward. This construction will allow us to write a supersymmetric lattice gauge theory as a zero dimensional matrix theory in terms of matrices satisfying the orbifold constraints. Thus, we will be able to maneuver from matrix to lattice or vice versa whenever needed. Then by simple manipulations, we show that the algebra of functions on the noncommutative discrete torus is the same as the algebra of ordinary functions on the commutative discrete torus where the usual product is deformed to the lattice Moyal $\star$ product. Most of the results of this section are known in the literature [43, 13, 30]. The results which will be used in subsequent sections are presented here for the clarity of discussion.

We start with $d = 0$ dimensional parent matrix theory with a $\mathrm{U}(N_c)$ gauge group, where $N_c = m_1 q m_2 q k$ is the number of colors and m_1, m_2, q, k are arbitrary integers. The integer k will serve as the rank of the gauge group of the target theory, $(m_1 q)^2$ will be the number of sites on the two dimensional lattice and m_2 will play some role in the noncommutativity parameter. In all of our examples, the parent matrix theories are the variations of zero dimensional SYM theories with $\mathcal{Q} = 4, 8, 16$ supersymmetries. The global nongauge symmetry group G_R has at least a $\mathrm{U}(1)^2$ subgroup which will be sufficient for our purpose. Any field in the parent matrix theory carries charges (r_1, r_2) under the $\mathrm{U}(1)^2$ which determines how they get projected. However, this section is not specific to supersymmetric theories. Any gauge field theory with adjoint matter on a noncommutative or commutative lattice can be obtained in the same way. Hence, we will try to minimize discussions related to supersymmetry until subsequent sections.

Let Φ be a generic matrix in the parent matrix theory and let us consider a $\mathbb{Z}_{m_2 q} \times \mathbb{Z}_{m_2 q}$ orbifold projection. The sub-blocks of the matrix Φ remaining invariant under the constraints

$$\Phi = \omega_{m_2 q}^{r_a} \Omega_a^\dagger \Phi \Omega_a \qquad a = 1, 2 \tag{2.1}$$

are said to satisfy the orbifold constraint and they form the field content of the orbifold theory. The phase $\omega_{m_2 q} = e^{2\pi i/(m_2 q)}$ is $m_2 q$-th primitive root of unity. The r_a are integer charges under an abelian subgroup of the global symmetry group G_R. The orbifold matrices $\Omega_a \in \mathrm{U}(N_c)$ are rank-N_c matrices forming a projective representation of the group $\mathbb{Z}_{m_2 q}$. A particular matrix representation for Ω_a may be written as

$$\begin{aligned} \Omega_1 &= (U_{m_1 m_2 q})^{m_1} \otimes (V_q^\dagger)^p \otimes \mathbf{1}_k \\ \Omega_2 &= (V_{m_1 m_2 q})^{m_1} \otimes U_q^\dagger \otimes \mathbf{1}_k \,, \end{aligned} \tag{2.2}$$

where the subscripts of $U, V, \mathbf{1}$ denote their rank and superscripts stand for the corresponding power. The U_q and V_q are called clock (position) and shift (translation) matrices and are given by

$$U_q = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{q-1} \end{pmatrix} \qquad V_q = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix} \tag{2.3}$$

and they satisfy the algebra

$$U_q V_q = \omega_q^{-1} V_q U_q \,. \tag{2.4}$$

We choose p and q as coprime for later convenience.

The orbifold matrices Ω_a may be thought as generators of the projective representation of the abelian group $\mathbb{Z}_{m_2 q} \times \mathbb{Z}_{m_2 q}$; each factor is embedded in $\mathrm{U}(N_c)$. Even though the group is abelian, the two sets of generators Ω_a's in eq. (2.2) do not commute with each other. Instead they satisfy

$$\Omega_1 \Omega_2 = e^{2\pi i \theta} \Omega_2 \Omega_1 \qquad \text{where} \qquad \theta = -\frac{m_1}{m_2 q} + \frac{p}{q} \qquad \text{modulo } \mathbb{Z} \tag{2.5}$$

and the rational number $\theta \in [0,1)$ which appears as the noncommutativity of Ω_a's is called the *discrete torsion*. It is not the noncommutativity parameter of the discrete torus on which the SYM theory lives, but it turns out to be closely related. There is a good reason that we have split the definition of θ into two pieces. In fact, as we will explain in detail in section 4, the second term in the sum may be thought of as due to a background magnetic flux.

In order to find the most general solution to the constraints eq. (2.1), we first introduce a set of matrices D_a which play two important roles: (i) They are displacement operators on the noncommutative lattice; (ii) They transform as if they carry a definite **r**-charge. We will first explain the second property and use it to *neutralize* the orbifold constraint [13, 30].[2]

The first property will be better appreciated when the real space basis for the lattice is introduced. In order to find solution to eq. (2.1), let us introduce $D_1, D_2 \in \mathrm{U}(N_c)$ as

$$\begin{aligned} D_1 &= V^\dagger_{m_1 m_2 q} \otimes \mathbf{1}_q \otimes \mathbf{1}_k \\ D_2 &= U_{m_1 m_2 q} \otimes \mathbf{1}_q \otimes \mathbf{1}_k \,. \end{aligned} \tag{2.6}$$

The remarkable property of these matrices is that D_1 transforms as if it has a charge vector $\mathbf{r} = (1,0)$ and similarly, D_2 transforms like it has a charge vector $(0,1)$. In general, $D_1^{r_1} D_2^{r_2}$ obeys the orbifold constraints that a field with charge vector $r_1(1,0) + r_2(0,1) = (r_1, r_2)$ would obey. Namely,

$$(D_1^{r_1} D_2^{r_2}) = \omega_{m_2 q}^{r_a} \, \Omega_a^\dagger (D_1^{r_1} D_2^{r_2}) \Omega_a \qquad a = 1,2\,. \tag{2.7}$$

[2] These matrices are also known as twist-eaters in the literature. But here we prefer not to call them that for two reasons. One is that they don't really serve that purpose here, and the other reason is that the real twist eating solutions are associated with 't Hooft's boundary conditions and will show up in section 4

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Since D_1 and D_2 can also be interpreted as finite translation operators in orthogonal directions in noncommutative lattice, they do not commute with each other. Instead, they satisfy the relation

$$D_1 D_2 = \omega_{m_1 m_2 q}^{-1} D_2 D_1 \,. \tag{2.8}$$

The main significance of D_a's may be realized as follows. Let Φ have $\mathbf{r}$-charge vector (r_1, r_2). Let us define $\widetilde{\Phi}$ by the equation

$$\Phi = \begin{cases} \widetilde{\Phi} D_1^{r_1} D_2^{r_2} & \text{if } r_1 > 0,\, r_2 > 0 \\ D_1^{r_1} \widetilde{\Phi} D_2^{r_2} & \text{if } r_1 < 0,\, r_2 > 0 \\ D_2^{r_2} D_1^{r_1} \widetilde{\Phi} & \text{if } r_1 < 0,\, r_2 < 0 \,. \end{cases} \tag{2.9}$$

It is easy to show that $\widetilde{\Phi}$ obeys the orbifold constraint as if it is neutral under the global symmetry group G_R, i.e, $\widetilde{\Phi}$ has charge vector $\mathbf{r} = (0,0)$. In that sense, eq. (2.9) is a neutralization procedure. By using eq. (2.1) and eq. (2.9), we obtain the neutral orbifold constraints for $\widetilde{\Phi}$ as

$$\widetilde{\Phi} = \Omega_a^\dagger \widetilde{\Phi} \Omega_a \qquad a = 1, 2 \,. \tag{2.10}$$

Once the neutral constraint is solved, we can immediately get the solution for the orbifold constraint eq. (2.1) by using the D_a matrices as in eq. (2.9) which follows easily from the invertibility of D_a matrices.

Noncommutative lattice basis and crossed product algebra. To identify the set of solutions to neutral orbifold constraint eq. (2.10) as the noncommutative lattice, we need to construct the explicit isomorphism between the lattice and the matrices satisfying eq. (2.10). The complete set of commutants of the orbifold matrices can be obtained by using the matrices $Z_a \in \mathrm{U}(m_1 q m_2 q)$

$$Z_1 = (U_{m_1 m_2 q})^{m_2} \otimes V_q^\dagger \,, \qquad Z_2 = (V_{m_1 m_2 q})^{m_2} \otimes (U_q^\dagger)^r \,. \tag{2.11}$$

The Z_a's are constructed such that $Z_a \otimes \mathbf{1}_k$ commutes with the orbifold matrices. The commutators $[Z_1 \otimes \mathbf{1}_k, \Omega_a] = [Z_2 \otimes \mathbf{1}_k, \Omega_2] = 0$ are trivially satisfied. The condition for $Z_2 \otimes \mathbf{1}_k$ to commute with Ω_1 requires $\frac{1-rp}{q} = s$ to be an integer. The equation (also known as a first order Diophantine equation)

$$1 = qs + pr \tag{2.12}$$

is guaranteed to have a solution (see [44, page 276]) for some integers (s, r) if the greatest common divisor of p and q is 1. Since we assumed p and q are co-prime, by construction there exists such a pair of integers (s, r).

The base matrices Z_a are cyclic with period $m_1 q$, and they satisfy $Z_a^{m_1 q} = \mathbf{1}_{m_1 q m_2 q}$. The Z_1 and Z_2 are projective representations of the abelian cyclic group $\mathbb{Z}_{m_1 q}$, and they satisfy

$$Z_1 Z_2 = e^{-2\pi i \theta'} Z_2 Z_1 \,, \qquad \theta' = +\frac{m_2}{m_1 q} - \frac{r}{q} \,, \tag{2.13}$$

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where the rational number $\theta' \in [0,1)$ is the dimensionless noncommutativity parameter of the lattice. The discrete torsion θ and noncommutativity θ' are related by an $\mathrm{SL}(2,\mathbb{Z})$ transformation

$$\theta' = \frac{r\theta + s}{-q\theta + p} \qquad \text{where} \qquad \begin{pmatrix} r & s \\ -q & p \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,. \tag{2.14}$$

The space of solutions to the orbifold condition is spanned by $(m_1 q)^2$ matrices

$$\{Z_1^{p_1} Z_2^{p_2} \,|\, p_1, p_2 = 1, \ldots, m_1 q\} \tag{2.15}$$

which form a complete and orthogonal basis for the lattice (for a proof, see [13]). This basis is not hermitean but can be made so by introducing a phase factor:

$$J_{\mathbf{p}} = e^{-\pi i \theta' p_1 p_2} Z_2^{p_2} Z_1^{p_1} \qquad \text{where} \qquad \mathrm{J}_{\mathbf{p}}^{\dagger} = \mathrm{J}_{-\mathbf{p}}\,. \tag{2.16}$$

The basis we have constructed will be identified as the momentum basis of the lattice. It has formally the same algebraic properties as the plane wave basis on a noncommutative space which satisfies

$$e^{ip_i \hat{x}^i} e^{iq_j \hat{x}^j} = e^{-\frac{i}{2} p_i \theta^{ij} q_j} e^{i(p_i + q_i)\hat{x}^i}\,, \tag{2.17}$$

where p_i and q_i label components of momenta and θ^{ij} is a real-valued antisymmetric matrix. Using the defining relation eq. (2.16) for the regularized momentum basis for the lattice, we obtain the corresponding regularized algebra as

$$J_{\mathbf{p}}\, J_{\mathbf{q}} = e^{-\pi i \theta' (p_1 q_2 - p_2 q_1)} J_{\mathbf{p}+\mathbf{q}} = e^{-\pi i \theta' \mathbf{p} \wedge \mathbf{q}} J_{\mathbf{p}+\mathbf{q}}\,. \tag{2.18}$$

The $\mathbf{p}$ labels momentum on a periodic lattice, in a two dimensional Brillouin zone $\mathbb{Z}_L^2$ where the addition of momenta is understood to be modulo $L = m_1 q$. The algebra in eq. (2.18) is a realization of a *crossed product algebra*. It is easy to see that this algebra is not commutative for generic θ', and it involves trigonometric functions as the structure constants: [45][3]

$$[J_{\mathbf{p}}\,, J_{\mathbf{q}}] = -2i\, \sin(\pi \theta' \mathbf{p} \wedge \mathbf{q})\, J_{\mathbf{p}+\mathbf{q}}\,. \tag{2.19}$$

The product is associative, i.e. $J_{\mathbf{p}}(J_{\mathbf{q}} J_{\mathbf{r}}) = (J_{\mathbf{p}} J_{\mathbf{q}}) J_{\mathbf{r}}$. The crossed product algebra of lattice generators gives the first glimpse of the appearance of noncommutativity of the base space in the target theory. We emphasize that the matrices $J_{\mathbf{p}}$ form a projective representation for the abelian group $\mathbb{Z}_{m_1 q} \times \mathbb{Z}_{m_1 q}$ which is in fact the lattice. Each point $\mathbf{p}$ on the momentum space lattice is represented by a unique matrix $J_{\mathbf{p}}$. For more on projective representation of finite groups, see [46].

In order to construct a noncommutative field theory with periodic boundary conditions, we need to restrict θ'. The periodicity conditions $J_{\mathbf{p}+L\hat{\mathbf{e}}_a} = J_{\mathbf{p}+L\hat{\mathbf{e}}_b} = J_{\mathbf{p}}$ are satisfied if $\theta' L p_a$ is an even integer for all $p_a = 1, \ldots, L$. This holds if

$$L\theta' = (-r m_1 + m_2) \in 2\mathbb{Z}\,. \tag{2.20}$$

[3]The structure constants of this algebra show up in the momentum space formulation of the action, in particular at cubic and quartic vertices. For example, this is the case in [11] where a momentum space formulation of continuum noncommutative SYM is given. The above algebra is a UV regulation of the one given in [11]. However, in the rest of this paper, we will mostly work in position space.

Now, we are ready to write down the most general configuration of the matrix $\widetilde{\Phi}$ satisfying the neutral orbifold constraint eq. (2.10) :

$$\widetilde{\Phi} = \sum_{\mathbf{p}\in\mathbb{Z}_L^2} J_{\mathbf{p}}^{\dagger} \otimes \Phi_{\mathbf{p}} \,, \tag{2.21}$$

where $\Phi_{\mathbf{p}}$ is a rank k field matrix that we associate to each momenta $\mathbf{p}$ in the Brillouin zone. The basis matrices $J_{\mathbf{p}}$ are rank m_1qm_2q and $\widetilde{\Phi}$ is rank m_1qm_2qk. The equation has to be understood as a map from the dual momentum lattice to a matrix symbol[4] and vice versa.

The coordinate basis for the lattice is the discrete Fourier transform of the momentum basis given by

$$\Delta_{\mathbf{n}} = \frac{1}{L^2} \sum_{\mathbf{p}\in\mathbb{Z}_L^2} J_{\mathbf{p}}\, \omega^{\mathbf{p}.\mathbf{n}} \,, \qquad J_{\mathbf{p}} = \sum_{\mathbf{n}\in\mathbb{Z}_L^2} \Delta_{\mathbf{n}}\, \omega^{-\mathbf{p}.\mathbf{n}} \,. \tag{2.22}$$

The index $\mathbf{n}$ spans a two dimensional real space lattice where each component is ranging from $1, \ldots L$. One can rewrite eq. (2.21) in a position space lattice as

$$\widetilde{\Phi} = \sum_{\mathbf{n}\in\mathbb{Z}_L^2} \Delta_{\mathbf{n}} \otimes \Phi_{\mathbf{n}}, \qquad \Phi_{\mathbf{n}} = \mathrm{Tr}\left(\widetilde{\Phi}(\Delta_{\mathbf{n}} \otimes \mathbf{1}_k)\right) , \tag{2.23}$$

where $\Phi_{\mathbf{n}}$ is a $k \times k$ field matrix that we associate with each site $\mathbf{n} \in \mathbb{Z}_L^2$ in the position space lattice. The second relation follows from the completeness and orthonormality of the noncommutative basis $\Delta_{\mathbf{n}}$. These two relations reveal that if we know the value of a field at each site $\mathbf{n}$ on the lattice, we can merge them into a single large matrix, the matrix symbol. Or equivalently, if we know the matrix symbol, we can unravel it to obtain the field matrix at each site of the lattice. Here, we list certain relations satisfied by basis matrices:[5]

$$\begin{aligned} \sum_{\mathbf{n}\in\mathbb{Z}_L^2} \Delta_{\mathbf{n}} &= \mathbf{1}_{m_1qm_2q} \,, \\ \mathrm{Tr}(\Delta_{\mathbf{n}}) &= 1 \,, \\ \mathrm{Tr}(\Delta_{\mathbf{n}}\Delta_{\mathbf{m}}) &= \delta_{\mathbf{n},\mathbf{m}} \,, \\ \mathrm{Tr}(\Delta_{\mathbf{k}}\Delta_{\mathbf{l}}\Delta_{\mathbf{m}}) &= \frac{1}{L^2}\, \omega^{-\frac{2}{L\theta'}(\mathbf{k}-\mathbf{m})\wedge(\mathbf{l}-\mathbf{m})} \,. \end{aligned} \tag{2.24}$$

The phase in the last formula is associated with a nonlocal cubic vertex. The counterpart of these relations in commutative case is the same except for the cubic and higher order vertices. For example, in the commutative case, the cubic vertex is

$$\mathrm{Tr}(\Delta_{\mathbf{k}}\Delta_{\mathbf{l}}\Delta_{\mathbf{m}}) = \delta_{\mathbf{k},\mathbf{m}}\, \delta_{\mathbf{l},\mathbf{m}} \,, \tag{2.25}$$

[4] The matrix $\widetilde{\Phi}$ defined as in eq. (2.21) is sometimes called the Weyl symbol or matrix symbol.

[5] The trace is rescaled by a factor $\frac{m_2}{m_1}$ so that $\mathrm{Tr}\,\mathbf{1}_{m_1qm_2q} = Volume(Lattice) = L^2$, the dimensionless volume of the lattice. With this convention, the identities for basis matrices eq. (2.24) are formally the same as the conventional continuum ones, see [7].

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where the locality is manifest. The meaning of the operators D_a in eq. (2.6) is also clear in the coordinate basis. They act as finite difference operators on the noncommutative lattice,

$$D_a(\Delta_{\mathbf{n}} \otimes \Phi_{\mathbf{n}})D_a^{-1} = \Delta_{\mathbf{n}-\hat{\mathbf{e}}_a} \otimes \Phi_{\mathbf{n}} \,.. \tag{2.26}$$

From noncommutative basis to Moyal $\star$ product. Next, we show that the algebra of the matrix valued functions on a noncommutative discrete torus can also be regarded as the usual algebra of matrices on the commutative discrete torus where the usual matrix product is replaced by the lattice Moyal $\star$ product. The construction here also provides an unambiguous definition of the $\star$ product on the lattice. Assume two matrices $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ obey the orbifold conditions eq. (2.10). Then so does their product $\widetilde{\Phi}_1\widetilde{\Phi}_2$, which may be expressed in the complete lattice basis as

$$\widetilde{\Phi}_1\widetilde{\Phi}_2 = \sum_{\mathbf{n}\in\mathbb{Z}_L^2} \Delta_{\mathbf{n}} \otimes [\Phi_{1,\mathbf{n}} \star \Phi_{2,\mathbf{n}}]\,, \tag{2.27}$$

where we define $[\Phi_{1,\mathbf{n}} \star \Phi_{2,\mathbf{n}}] = \mathrm{Tr}\left(\widetilde{\Phi}_1\widetilde{\Phi}_2(\Delta_{\mathbf{n}} \otimes \mathbf{1}_k)\right)$. To obtain the expression for $\star$ product, consider

$$\widetilde{\Phi}_1\widetilde{\Phi}_2 = \sum_{\mathbf{n},\mathbf{m}\in\mathbb{Z}_L^2} \Delta_{\mathbf{n}}\Delta_{\mathbf{m}} \otimes \Phi_{1,\mathbf{n}}\Phi_{2,\mathbf{m}}\,, \tag{2.28}$$

where $\Phi_{1,\mathbf{n}}\Phi_{2,\mathbf{m}}$ is the usual matrix product at this stage. One can calculate the product of basis matrices $\Delta_{\mathbf{n}}\Delta_{\mathbf{m}}$ by using eq. (2.22) and the definition of the crossed product algebra eq. (2.18) :

$$\Delta_{\mathbf{n}}\Delta_{\mathbf{m}} = \frac{1}{L^2}\sum_{\mathbf{s}\in\mathbb{Z}_L^2}{}' \Delta_{\mathbf{s}}\ \omega^{-\frac{2}{L\theta'}(\mathbf{m}-\mathbf{s})\wedge(\mathbf{n}-\mathbf{s})} \tag{2.29}$$

where the prime on the summation sign indicates that the sum is over $\mathbf{s} \in \mathbb{Z}_L^2$ such that either

$$(\mathbf{m}-\mathbf{s}) \in \frac{L\theta'}{2}\mathbb{Z} \times \frac{L\theta'}{2}\mathbb{Z} \qquad \text{or} \qquad (\mathbf{n}-\mathbf{s}) \in \frac{L\theta'}{2}\mathbb{Z} \times \frac{L\theta'}{2}\mathbb{Z} \tag{2.30}$$

modulo periodicity of the lattice. Substituting this result into eq. (2.28) and using orthonormality of the basis, we obtain the definition of the Moyal $\star$ product on the lattice. We have

$$[\Phi_{1,\mathbf{s}} \star \Phi_{2,\mathbf{s}}] = \frac{1}{L^2}\sum_{\mathbf{n},\mathbf{m}\in\mathbb{Z}_L^2}{}' \Phi_{1,\mathbf{n}}\Phi_{2,\mathbf{m}}\,\omega^{-\frac{2}{L\theta'}(\mathbf{m}-\mathbf{s})\wedge(\mathbf{n}-\mathbf{s})}\,, \tag{2.31}$$

where the product of matrices within the sum is the usual matrix product. The prime on the double sum is to remind us that one of the sums is over all the lattice points, whereas the other sum is over the sublattice eq. (2.30) modulo periodicity $L\times L$. This shows that the algebra of functions on the noncommutative discrete torus can be traded with a deformed algebra of functions on a commutative discrete torus. The deformed product eq. (2.31) is called the Moyal $\star$ product and it is nonlocal. In particular, for the minimum nontrivial value of θ'

$$\theta'_{\min} = \frac{2}{L} = \frac{2}{m_1 q} \tag{2.32}$$

both sums run over all sites on the lattice with an oscillatory kernel.

As a last remark on the lattice Moyal $\star$ product eq. (2.31), we observe that the $\star$ product of a constant matrix field M on the lattice with a varying matrix field $\Phi_{1,\mathbf{s}}$ is just the usual multiplication,

$$\Phi_{1,\mathbf{s}} \star M = \Phi_{1,\mathbf{s}} M\,. \tag{2.33}$$

In particular, $\star$ multiplication of two constant matrices M and N is just the usual multiplication $M \star N = MN$.

3. Regularization for noncommutative $\mathcal{N} = (2,2)$ SYM in $d = 2$

The action of the parent theory is a variant of dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with $\mathrm{U}(N_c)$ gauge group from $d = 4$ Euclidean dimensions to zero dimension. The standard reduction of $\mathcal{N} = 1$ SYM possess a $\mathcal{Q} = 4$ supersymmetry and $\mathrm{SO}(4) \times \mathrm{U}(1)$ global R-symmetry. The $\mathrm{SO}(4)$ symmetry is inherited from the Lorentz symmetry and $\mathrm{U}(1)$ is the global R-symmetry of the Euclidean theory prior to reduction.

The zero dimensional matrix model is composed of bosonic and fermionic Lie algebra valued matrices. The bosonic matrices are $\{z_1, z_2, \bar{z}_1, \bar{z}_2, d\}$ where z_a $(\bar{z}_a)$ are the complexifications of dimensional reduction of the 4-vector gauge potential and d is an auxiliary field. The fermionic matrices are Grassmann valued matrices $\{\psi_1, \psi_2, \lambda, \xi\}$ which were components of two independent, complex two-component spinors in four dimensions prior to dimensional reduction.

We deform the reduced matrix model by a pure phase ζ, in a way that the action only respects one out of four supersymmetries and no other. We call this parent theory the *deformed matrix model*. The action is

$$S_\zeta = \frac{1}{g^2}\,\mathrm{Tr}\left[\frac{1}{2}d^2 + id[\bar{z}_a, z_a] + 2\big|[z_1, z_2]_\zeta\big|^2 + \sqrt{2}\,(\lambda\,[\bar{z}_a, \psi_a] + \xi\,[z_1, \psi_2]_\zeta + \xi\,[\psi_1, z_2]_\zeta)\right], \tag{3.1}$$

where $[z_1, z_2]_\zeta \equiv (z_1 z_2 - \zeta z_2 z_1)$ etc. The deformed matrix model action eq. (3.1) has all $\mathcal{Q} = 4$ supersymmetries for $\zeta = 1$ and has only $\mathcal{Q} = 1$ for $\zeta \neq 1$. For $\zeta = 1$, after eliminating the auxiliary field d, the bosonic part of the action is the zero dimensional counterpart of the sum of electric and magnetic energies, whereas the fermionic part is the interaction term inherited from the covariant derivative.

The supersymmetry that the action respects may be written as

$$\begin{aligned}
\delta z_a &= \sqrt{2} i\eta\psi_a \qquad a = 1,2\\
\delta \bar{z}_a &= 0\\
\delta \psi_a &= 0\\
\delta \lambda &= \eta d\\
\delta \xi &= -2i\eta[\bar{z}_2, \bar{z}_1]_{\zeta^*}\\
\delta d &= 0\,.
\end{aligned} \tag{3.2}$$

where η is a Grassmann valued parameter. The double variation of any field by δ is identically zero. With $\delta = i\eta Q = i\eta\partial_\theta$, the transformation laws can be realized in terms of

	$\mathbf{Z}_1$	$\overline{z}_1$	$\mathbf{Z}_2$	$\overline{z}_2$	$\mathbf{\Lambda}$	$\mathbf{\Xi}_{ab}$
r_1	$+1$	-1	0	0	0	-1
r_2	0	0	$+1$	-1	0	-1

Table 1: The $r_{1,2}$ charges of the fields of the $\mathcal{Q}=1$ deformed matrix theory which define the orbifold projection.

the following $\mathcal{Q}=1$ superfields:

$$\begin{aligned}\mathbf{Z}_a &= z_a + \sqrt{2}\theta\psi_a \\ \overline{z}_a &= \overline{z}_a \\ \mathbf{\Lambda} &= \lambda - i\theta d \\ \mathbf{\Xi} &= \xi - 2\theta[\overline{z}_2,\overline{z}_1]_{\zeta^*}\,. \end{aligned} \tag{3.3}$$

Note that the $\overline{z}_a$ are singlets under supersymmetry. The action of the deformed matrix model in manifestly $\mathcal{Q}=1$ supersymmetric form may be written as

$$S_\zeta = \frac{1}{g^2}\,\mathrm{Tr}\int d\theta\left[-\frac{1}{2}\mathbf{\Lambda}\partial_\theta\mathbf{\Lambda} - \mathbf{\Lambda}\left[\overline{z}_a,\mathbf{Z}_a\right] - \mathbf{\Xi}\;[\mathbf{Z}_1,\mathbf{Z}_2]_\zeta\right]. \tag{3.4}$$

The deformed matrix model has a global symmetry group with a subgroup $G_R = \mathrm{U}(1)\times\mathrm{U}(1)$. We employ G_R to orbifold the theory as in [20]. We assign the R-charges as in table 1.

Any of the fields of the deformed matrix model (with an $\mathbf{r}$-charge vector given in table 1) satisfying the orbifold constraint eq. (2.1) may be written as in eq. (2.9) where the fields satisfying the neutral constraint are expressed as in eq. (2.23). For example, the superfield matrix $\mathbf{Z}_1$ with r-charge vector $(1,0)$ satisfying orbifold constraints eq. (2.1) is written as

$$\mathbf{Z}_1 = \widetilde{\mathbf{Z}}_1 D_1 = \left(\sum_{\mathbf{n}\in\mathbb{Z}_L^2}\Delta_{\mathbf{n}}\otimes\mathbf{Z}_{1,\mathbf{n}}\right)D_1 \tag{3.5}$$

and a similar expression can be written for the other matrices as well. The $k\times k$ matrix field $\mathbf{Z}_{1,\mathbf{n}}$ is associated with the unit cell $\mathbf{n}\in\mathbb{Z}_L^2$. To understand where exactly a field resides on the unit cell, we need to examine its gauge transformation properties. Consider again for simplicity $\mathbf{Z}_1$. Under a gauge transformation, $\mathbf{Z}_1\to\widetilde{g}\mathbf{Z}_1\widetilde{g}^\dagger$ or equivalently

$$\widetilde{\mathbf{Z}}_1\to\widetilde{g}\widetilde{\mathbf{Z}}_1(D_1\widetilde{g}^\dagger D_1^{-1}) \tag{3.6}$$

where

$$\widetilde{g} = \sum_{\mathbf{n}\in\mathbb{Z}_L^2}\Delta_{\mathbf{n}}\otimes g_{\mathbf{n}} \tag{3.7}$$

is the matrix symbol for gauge rotation. Using eq. (2.26) and eq. (2.24) and by taking the trace of both sides of eq. (3.6), we obtain

$$\mathbf{Z}_{1,\mathbf{n}}\to g_{\mathbf{n}}\star\mathbf{Z}_{1,\mathbf{n}}\star g^\dagger_{\mathbf{n}+\hat{\mathbf{e}}_1} \tag{3.8}$$

the star gauge transformation of $\mathbf{Z}_{1,\mathbf{n}}$ where $g_{\mathbf{n}}$ is star unitary gauge rotation matrix. Similarly, we have $\bar{z}_{1,\mathbf{n}} \to g_{\mathbf{n}+\hat{\mathbf{e}}_1} \star \bar{z}_{1,\mathbf{n}} \star g^\dagger_{\mathbf{n}}$. Both $\mathbf{Z}_{1,\mathbf{n}}$ and $\bar{z}_{1,\mathbf{n}}$ are residing on the link between sites $(\mathbf{n}, \mathbf{n}+\hat{\mathbf{e}}_1)$ and transforming oppositely under star gauge transformation. Our choice of the ordering in eq. (2.9) was in fact motivated by this convention. The orientation of the link variables are determined by their $\mathbf{r}$-charges. To summarize, the star gauge transformation properties of the lattice fields are

$$\begin{aligned}
\mathbf{Z}_{a,\mathbf{n}} &\to g_{\mathbf{n}} \star \mathbf{Z}_{a,\mathbf{n}} \star g^\dagger_{\mathbf{n}+\hat{\mathbf{e}}_a} \\
\bar{z}_{a,\mathbf{n}} &\to g_{\mathbf{n}+\hat{\mathbf{e}}_a} \star \bar{z}_{a,\mathbf{n}} \star g^\dagger_{\mathbf{n}} \\
\mathbf{\Lambda}_{\mathbf{n}} &\to g_{\mathbf{n}} \star \Lambda_{\mathbf{n}} \star g^\dagger_{\mathbf{n}} \\
\mathbf{\Xi}_{\mathbf{n}} &\to g_{\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2} \star \mathbf{\Xi}_{\mathbf{n}} \star g^\dagger_{\mathbf{n}} \,.
\end{aligned} \tag{3.9}$$

From the supersymmetry transformation of the deformed matrix theory eq. (3.2), one can extract the supersymmetry transformation of the lattice theory by using the completeness and orthogonality eq. (2.24) of the basis $\Delta_{\mathbf{n}}$ and the algebra eq. (2.7) of translation D_a matrices. In terms of individual components, the supersymmetry transformations of the lattice fields can be found as

$$\begin{aligned}
\delta z_{a,\mathbf{n}} &= i\sqrt{2}\,\eta\psi_{a,\mathbf{n}} \\
\delta \bar{z}_{a,\mathbf{n}} &= 0 \\
\delta \psi_{a,\mathbf{n}} &= 0 \\
\delta \xi_{\mathbf{n}} &= -2i\eta(\bar{z}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} \star \bar{z}_{1,\mathbf{n}} - \zeta^* \,\omega^{-1}_{m_1 m_2 q}\; \bar{z}_{1,\mathbf{n}+\hat{\mathbf{e}}_2} \star \bar{z}_{2,\mathbf{n}}) \\
\delta \lambda_{\mathbf{n}} &= \eta d_{\mathbf{n}} \\
\delta d_{\mathbf{n}} &= 0\,,
\end{aligned} \tag{3.10}$$

where the extra phase factor $\omega_{m_1 m_2 q}$ arises because of noncommutativity of translations in orthogonal directions. Note that $\bar{z}_{a,\mathbf{n}}$ are supersymmetric singlets. These transformation laws lead to supersymmetric multiplets

$$\begin{aligned}
\mathbf{\Lambda}_{\mathbf{n}} &= \lambda_{\mathbf{n}} - i\theta d_{\mathbf{n}}\,, \\
\mathbf{Z}_{a,\mathbf{n}} &= z_{a,\mathbf{n}} + \sqrt{2}\,\theta\,\psi_{a,\mathbf{n}}\,, \\
\mathbf{\Xi}_{\mathbf{n}} &= \xi_{\mathbf{n}} - 2\theta\;(\bar{z}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} \star \bar{z}_{1,\mathbf{n}} - \zeta^* \,\omega^{-1}_{m_1 m_2 q}\; \bar{z}_{1,\mathbf{n}+\hat{\mathbf{e}}_2} \star \bar{z}_{2,\mathbf{n}})\,.
\end{aligned} \tag{3.11}$$

Setting $\zeta\,\omega_{m_1 m_2 q} = 1$, we express the lattice action for the noncommutative $\mathcal{N} = (2,2)$ theory in manifestly $\mathcal{Q} = 1$ supersymmetric form as

$$\begin{aligned}
S = \frac{1}{g^2_{nc}} \sum_{\mathbf{n}\in\mathbb{Z}^2_L} \mathrm{tr} \int d\theta \Big[&-\frac{1}{2}\mathbf{\Lambda}_{\mathbf{n}} \star \partial_\theta \mathbf{\Lambda}_{\mathbf{n}} - \mathbf{\Lambda}_{\mathbf{n}} \star (\bar{z}_{a,\mathbf{n}-\hat{\mathbf{e}}_a} \star \mathbf{Z}_{a,\mathbf{n}-\hat{\mathbf{e}}_a} - \mathbf{Z}_{a,\mathbf{n}} \star \bar{z}_{a,\mathbf{n}}) - \\
&- \mathbf{\Xi}_{\mathbf{n}} \star (\mathbf{Z}_{1,\mathbf{n}} \star \mathbf{Z}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - \mathbf{Z}_{2,\mathbf{n}} \star \mathbf{Z}_{1,\mathbf{n}+\hat{\mathbf{e}}_2}) \Big]\,,
\end{aligned} \tag{3.12}$$

where $g^2_{nc} = g^2 \frac{m_1}{m_2}$ is the dimensionful coupling [6] Substituting the superfields and integrating over the Grassmann coordinate, we can obtain the action in components.

[6] In our normalization, the mass dimension for fields, coupling constant and Grassmann coordinate are:

$$[z] = [\bar{z}] = [\mathfrak{a}^{-1}] = 1\,, \quad [d] = 2\,, \qquad [\text{fermions}] = 3/2\,, \qquad [g^2_{nc}] = 4\,, \qquad [\partial_\theta] = -[\theta] = 1/2\,.$$

The action has a classical moduli space (flat directions in the field configuration along which the bosonic part of the action vanishes) for $\zeta = \omega^{-1}_{m_1 m_2 q}$ analogous to its commutative counterpart [20]. For any other value of ζ, there is generically no moduli space along which both moduli fields are nonvanishing, and a sensible continuum limit does not exist. We will examine these cases in subsection 3.2. The flatness condition can be written as

$$\begin{aligned} \mathrm{d}_{\mathbf{n}} &= \overline{z}_{a,\mathbf{n}-\hat{\mathbf{e}}_a} \star z_{a,\mathbf{n}-\hat{\mathbf{e}}_a} - z_{a,\mathbf{n}} \star \overline{z}_{a,\mathbf{n}} = 0 \\ z_{1,\mathbf{n}} &\star z_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - z_{2,\mathbf{n}} \star z_{1,\mathbf{n}+\hat{\mathbf{e}}_2} = 0 \end{aligned} \tag{3.13}$$

which are analogs of the D-flatness and F-flatness conditions that show up in supersymmetric gauge theories in higher dimensions.

As in the commutative counterpart, we consider a particular configuration in moduli space respecting the U(k) global symmetry, namely

$$z_{a,\mathbf{n}} = \overline{z}_{a,\mathbf{n}} = \frac{1}{\mathfrak{a}\sqrt{2}} \mathbf{1}_k \qquad a = 1, 2\,, \tag{3.14}$$

where $\mathfrak{a}$ is a parameter with dimension of length. We expand the action about the point eq. (3.14) by identifying the lattice spacing with $\mathfrak{a}$.

3.1 Continuum Limits

In any noncommutative field theory the quadratic part of the action must be exactly the same as the quadratic part of its commutative counterpart, both for the lattice and the continuum, which amounts to saying that the free propagators are the same. The difference starts with the momentum dependent phase factors attached to interaction vertices. Hence, our analysis of the tree level is quite similar to [20] and in fact, the identification of continuum fields in terms of lattice fields is exactly the same. Here, our emphasis will be on different characteristics of the continuum limit of noncommutative theories with respect to their commutative counterparts.

To construct the possible continuum limits of the non-commutative SYM theories, we first obtain the continuum Moyal $\star$ product from the lattice $\star$ product eq. (2.31) and identify the dimensionful noncommutativity parameter. Let $\mathbf{w} = \mathbf{s}\mathfrak{a}$, $\mathbf{x} = \mathbf{m}\mathfrak{a}$, $\mathbf{y} = \mathbf{n}\mathfrak{a}$ be dimensionful length. We need to replace the summation in eq. (2.31) with integration. But naively replacing the two summations with two integrations would simply be wrong since the summations are not over all lattice points. We can take without loss of generality one of the summations to be over all lattice points. Then we substitute $\sum_{\mathbf{m}} \mathfrak{a}^2 \to \int d^2x$. In the second summation, $\mathbf{n}$ runs over the set $\frac{L\theta'}{2}\mathbb{Z} \times \frac{L\theta'}{2}\mathbb{Z}$, a sublattice of integers, modulo periodicity of the lattice. Hence, the sum with a prime has to be replaced with

$$\sum_{\mathbf{n}\in\mathbb{Z}_L^2}{}' \mathfrak{a}^2 \to \left(\frac{2}{L\theta'}\right)^2 \int d^2y\,. \tag{3.15}$$

With these deliberations in mind, we do obtain the continuum Moyal star product from the lattice star product eq. (2.31) as

$$\Phi_1(\mathbf{w}) \star \Phi_2(\mathbf{w}) = \frac{1}{L^2\mathfrak{a}^4}\frac{4}{L^2\theta'^2}\int\int d^2x\, d^2y\ \Phi_1(\mathbf{x})\,\Phi_2(\mathbf{y})\ e^{-2i\frac{2\pi}{L^2\theta'\mathfrak{a}^2}(\mathbf{x}-\mathbf{w})\wedge(\mathbf{y}-\mathbf{w})} \tag{3.16}$$

$$= \int\int d^2x\, d^2y\ \Phi_1(\mathbf{x})\,\Phi_2(\mathbf{y})\ K(\mathbf{x}-\mathbf{w},\mathbf{y}-\mathbf{w})\,, \tag{3.17}$$

where the kernel of the integral is defined as

$$K(\mathbf{x}-\mathbf{w},\mathbf{y}-\mathbf{w}) = \frac{1}{\pi^2 \det\Theta'}e^{-2i(\mathbf{x}-\mathbf{w})_i(\Theta'^{-1})_{ij}(\mathbf{y}-\mathbf{w})_j}\,. \tag{3.18}$$

We identify the dimensionful noncommutativity parameter Θ'_{ij} as

$$\Theta'_{ij} = \frac{L^2\mathfrak{a}^2\theta'}{2\pi}\epsilon_{ij}\,, \tag{3.19}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$.

Finite volume. There are two types of continuum limit that one can consider with a finite volume and finite two dimensional coupling $g_2^2 = g_{nc}^2\mathfrak{a}^2$. One choice is finite noncommutativity and the other is zero noncommutativity. From the relation eq. (3.19), we observe that infinite noncommutativity is not compatible with the finite volume limit because $\theta' \in [0,1)$. Let α be a parametrically small number. Consider the scaling

$$\mathfrak{a} \sim \alpha, \qquad L \sim \frac{1}{\alpha}, \qquad g_{nc}^2 \sim \frac{1}{\alpha^2}\,, \tag{3.20}$$

which yields finite volume and finite two dimensional coupling. But there are two distinct outcomes for dimensionful noncommutativity parameter Θ' depending on the scaling of the dimensionless parameter θ':

$$\mathrm{Vol} = (\mathfrak{a}L)^2 \sim 1, \quad g_2^2 \sim 1, \quad \Theta' \sim \begin{cases} \alpha \ \text{ for } \theta' \sim \alpha \\ 1 \ \text{ for } \theta' \sim 1 \end{cases}. \tag{3.21}$$

The θ' on the lattice is defined modulo $\mathbb{Z}$, it is a rational number between zero and one with a minimum nontrivial value $\theta' = \frac{2}{L}$ and a generic value as in eq. (2.13). If θ' is kept fixed as we take the limit of large L as in eq. (3.20), we obtain a finite noncommutativity parameter Θ' in the continuum. If θ' vanishes in the limit eq. (3.20), then we obtain zero noncommutativity parameter in the continuum with fixed volume.

Infinite volume. The continuum limit in which the volume is taken to infinity is obtained from the scaling

$$\mathfrak{a} \sim \alpha, \qquad L \to \frac{1}{\alpha^2}, \qquad g_{nc}^2 \to \frac{1}{\alpha^2}\,. \tag{3.22}$$

Hence the continuum parameters and volume scale as

$$\mathrm{Vol} \sim \frac{1}{\alpha^2}, \qquad g_2^2 \sim 1, \qquad \Theta' \sim \begin{cases} 1 \quad \text{for } \theta' \sim \alpha^2 \\ \dfrac{1}{\alpha^2} \quad \text{for } \theta' \sim 1 \end{cases}. \tag{3.23}$$

Notice that the limit $\theta' \to 0$ which leads to zero noncommutativity parameter in finite volume leads to a finite dimensionful noncommutativity in the infinite volume. Similarly, the scaling $\theta' \sim 1$ which leads to the finite noncommutativity on finite volume leads to infinite dimensionful noncommutativity in the infinite volume because of the relation eq. (3.19).

Continuum: tree level. By expanding the action eq. (3.12) about the point eq. (3.14) in moduli space, we obtain the action of the continuum noncommutative $\mathcal{N} = (2, 2)$ SYM theory. Splitting the fluctuations of the link field into its hermitean and antihermitean parts, we have

$$z_a = \frac{1}{\sqrt{2}\mathfrak{a}}\mathbf{1}_k + \frac{s_a + iv_a}{\sqrt{2}}\,, \tag{3.24}$$

where s_a and v_a are hermitean $k \times k$ matrices. The fields s_a are identified as the scalar in the continuum, and v_a are the gauge fields. The fermions residing on the links and sites combine to form the Dirac spinors of the two dimensional theory. These results follows easily by using the property eq. (2.33) of the star product. Essentially, the analysis of the quadratic part of the action in the noncommutative theory reduces to the one in ordinary theory. The spectrum of the fermions is the same as the spectrum of scalars owing to exact supersymmetry on the lattice, and there are no doublers in the formulation [20].

The continuum star gauge transformations follow from the smooth gauge transformation on the lattice. Consider for example the transformation of $\mathbf{Z}_{a,\mathbf{n}}$ on the lattice eq. (3.9). By expanding the gauge transformation at $\mathbf{n} + \hat{\mathbf{e}}_a$ as $g(\mathbf{x} + \hat{\mathbf{e}}_a\mathfrak{a}) = g(\mathbf{x}) + \mathfrak{a}\partial_a g(\mathbf{x}) + O(\mathfrak{a}^2)$ and using star unitarity $g_{\mathbf{n}} \star g^\dagger_{\mathbf{n}} = 1$, we obtain the desired gauge transformation in the continuum

$$\begin{aligned} s_a(\mathbf{x}) &\to g(\mathbf{x}) \star s_a(\mathbf{x}) \star g^\dagger(\mathbf{x}) + O(\mathfrak{a}), \\ v_a(\mathbf{x}) &\to g(\mathbf{x}) \star v_a(\mathbf{x}) \star g^\dagger(\mathbf{x}) - ig(\mathbf{x}) \star \partial_a g^\dagger(\mathbf{x}) + O(\mathfrak{a}) \end{aligned} \tag{3.25}$$

where the scalars transform homogeneously and the vector potential transforms inhomogeneously. Similarly, the fermionic matter transforms in the adjoint representation of the group $\mathrm{U}(k)$ up to $O(\mathfrak{a})$ corrections.

The action of the continuum $\mathcal{N} = (2, 2)$ SYM target theory with gauge group $\mathrm{U}(k)$ may be written as

$$S = \frac{1}{g_2^2} \int d^2x \;\mathrm{tr}\left(D_m s_a \star D_m s_a + \frac{1}{2} v_{12} \star v_{12} - \frac{1}{2}(s_1 \star s_2 - s_2 \star s_1)^2 + \text{fermions} \right), \tag{3.26}$$

where $D_m s_a = \partial_m s_a + i(v_m \star s_a - s_a \star v_m)$ is the star covariant derivative of scalar s_a, and $v_{12} = \partial_1 v_2 - \partial_2 v_1 + i(v_1 \star v_2 - v_2 \star v_1)$ is the field strength. An important distinction between the noncommutative and ordinary SYM theory is the fact that the U(1) sector in the noncommutative theory is an interacting field theory and does not decouple from the $\mathrm{SU}(k)$ sector [47]. There are both three photon, three gluon as well as one photon, two gluon vertices in a perturbative expansion. We will discuss the continuum of the quantum theory after discussing the $\mathrm{SL}(2, \mathbb{Z})$ duality of a noncommutative gauge theory to a commutative gauge theory with flux.

3.2 The moral for the deformation

In order to reach the lattice action for noncommutative field theory, we started with a deformed matrix action with an a priori undetermined parameter ζ. Then we set $\zeta\omega_{m_1m_2q} = 1$ to obtain the lattice action for the noncommutative theory. Here, we will argue that $\zeta = \omega^{-1}_{m_1m_2q}$ is the only choice which yields a useful moduli space in the sense that it supports the lattice action for noncommutative $\mathcal{N} = (2,2)$ SYM and possesses a sensible continuum.

For an arbitrary value of the deformation parameter ζ, the second equation determining moduli space in eq. (3.13) would take the form

$$z_{1,\mathbf{n}} \star z_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - \zeta\omega_{m_1m_2q}\, z_{2,\mathbf{n}} \star z_{1,\mathbf{n}+\hat{\mathbf{e}}_2} = 0\,, \tag{3.27}$$

where $z_{a,\mathbf{n}}$ are $k \times k$ matrices and $\omega_{m_1m_2q}$ is (m_1m_2q)-th root of unity. There is a priori no relation between the integers m_1m_2q and k.

For generic values of k and (m_1m_2q), consider the $\mathbf{n}$-independent configurations of moduli fields satisfying the matrix equation eq. (3.27). By using eq. (2.33), we can rewrite the moduli equation as

$$z_1z_2 - \zeta\omega_{m_1m_2q}\, z_2z_1 = 0\,, \tag{3.28}$$

where z_a is a rank k, $\mathbf{n}$-independent matrix. Taking trace of both sides, we obtain

$$(1 - \zeta\omega_{m_1m_2q})\,\mathrm{tr}(z_1z_2) = 0\,. \tag{3.29}$$

If $\mathrm{tr}(z_1z_2) \neq 0$, the only way to have a solution is the vanishing of the first term, requiring $\zeta\omega_{m_1m_2q} = 1$.

For generic values of the parameter $\zeta\omega_{m_1m_2q}$, there is no solution at all to matrix equations eq. (3.28) unless one of the $z_a = 0$. That means one of the moduli fields is frozen at the origin of the moduli space and can not move. But in order to take the continuum limit, we have to take both $\langle z_{1,\mathbf{n}}\rangle$ and $\langle z_{2,\mathbf{n}}\rangle$ to large values of moduli (which corresponds to small lattice spacing) proportional to the identity. This can not be achieved under the given circumstances.

For some special values of parameters, say $\zeta = 1$ and $k = (m_1m_2q)$, there is a moduli space where both z_a are nonvanishing and satisfying $\mathrm{tr}(z_1z_2) = 0$. But it is a noncommutative moduli space and the identity matrix is not part of it. The noncommutative moduli space can be realized with the matrices

$$z_1 = \frac{1}{\mathfrak{a}}U_k, \qquad z_2 = \frac{1}{\mathfrak{a}}V_k\,, \tag{3.30}$$

where U_k and V_k are defined in eq. (2.3) and $\mathfrak{a}$ is a parameter with inverse mass dimension. Expanding the lattice action around such a background configuration does not lead to a sensible continuum action. For example, it contains terms such as

$$\sum_{\mathbf{n}} \mathrm{tr}\left|\frac{1}{\mathfrak{a}}(U_k \star z_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - \omega_k z_{2,\mathbf{n}} \star U_k)\right|^2 . \tag{3.31}$$

Even for smooth configurations of the fields $z_{2,\mathbf{n}}$, this term (as well as other terms in the action) tends to infinity as we take $\mathfrak{a} \to 0$ limit. Thus, even though there is a noncommutative moduli space, there is no sensible continuum limit that can emerge anywhere in this moduli space. Moreover, even at finite lattice spacing, we know that the free spectrum of a noncommutative theory has to be identical to its commutative counterpart. This is not realized within the noncommutative moduli space.

With the deformation we performed to the parent matrix theory, we do obtain a lattice action for noncommutative SYM theory which possesses a sensible continuum limit. In fact, the free spectrum of the noncommutative theory turns out to be identical to its ordinary counterpart. This is expected since the difference between the two theories at the perturbative level starts with momentum dependent phase associated to the cubic and higher order interaction vertices. Thus, the perturbative spectrum of the $\mathcal{N} = (2, 2)$ noncommutative SYM coincides with the ordinary SYM examined in [20].

4. Nonperturbative T-duality (Morita equivalence)

In section 3, we constructed a nonperturbative regularization for noncommutative $\mathcal{N} = (2, 2)$ supersymmetric gauge theories on torus. In this section, we will nonperturbatively prove that nontrivial topological sectors (with an integer background nonabelian magnetic flux) of a $\mathrm{U}(k)$ $\mathcal{N} = (2, 2)$ SYM theory on a commutative torus are equivalent to a noncommutative SYM theory on a larger size torus with a reduced gauge group. The noncommutativity parameter is uniquely determined by the integer magnetic flux. Our proof also implies that noncommutative SYM theories can be regularized by means of an ordinary SYM theory with 't Hooft flux. The discussion in this section is not specific to $\mathcal{N} = (2, 2)$, it also applies to $\mathcal{N} = (4, 4)$ and $\mathcal{N} = (8, 8)$ theories.

Let us start with the lattice action for a $\mathrm{U}(k)$ $\mathcal{N} = (2, 2)$ supersymmetric gauge theory on an $L \times L$ lattice. The action for this theory may be written as in [20]

$$S = \frac{1}{g^2} \sum_{\mathbf{n} \in \mathbb{Z}_L^2} \mathrm{tr}_{(k)} \int d\theta \Big[-\frac{1}{2} \boldsymbol{\Lambda}_{\mathbf{n}} \partial_\theta \boldsymbol{\Lambda}_{\mathbf{n}} - \boldsymbol{\Lambda}_{\mathbf{n}} \left(\overline{z}_{a,\mathbf{n}-\hat{\mathbf{e}}_a} \mathbf{Z}_{a,\mathbf{n}-\hat{\mathbf{e}}_a} - \mathbf{Z}_{a,\mathbf{n}} \overline{z}_{a,\mathbf{n}} \right) - \\ - \boldsymbol{\Xi}_{\mathbf{n}} \left(\mathbf{Z}_{1,\mathbf{n}} \mathbf{Z}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - \mathbf{Z}_{2,\mathbf{n}} \mathbf{Z}_{1,\mathbf{n}+\hat{\mathbf{e}}_2} \right) \Big] \tag{4.1}$$

or equivalently as a matrix theory action

$$S = \frac{1}{g^2} \mathrm{Tr}_{(L^2)} \mathrm{tr}_{(k)} \int d\theta \left[-\frac{1}{2} \widetilde{\boldsymbol{\Lambda}} \partial_\theta \widetilde{\boldsymbol{\Lambda}} - \widetilde{\boldsymbol{\Lambda}} [D_a^{-1} \widetilde{\overline{z}}_a,\ \widetilde{\mathbf{Z}}_a D_a] - \widetilde{\boldsymbol{\Xi}} [\widetilde{\mathbf{Z}}_1 (D_1 \widetilde{\mathbf{Z}}_2 D_1^{-1}) - \mathbf{Z}_2 (D_2 \widetilde{\mathbf{Z}}_1 D_2^{-1})] \right], \tag{4.2}$$

where we have used the commuting rank-L^2 basis matrices to express the lattice as a large matrix, and D_a are the commuting finite difference operators. The matrices with tilde are rank-$L^2 k$ and are defined as in eq. (2.23). The $\mathrm{Tr}_{(L^2)}$ is the trace over the lattice basis and $\mathrm{tr}_{(k)}$ is over the $\mathrm{U}(k)$ gauge group. The subscripts of trace denote the rank of the corresponding matrices.

To construct the different topological sectors of the theory, we impose on the superfields the following twisted boundary conditions [41, 40]

$$\begin{aligned}
\mathbf{\Lambda}_{\mathbf{n}+L\hat{\mathbf{e}}_a} &= \Gamma_{a,\mathbf{n}}\ \mathbf{\Lambda}_{\mathbf{n}}\ \Gamma^\dagger_{a,\mathbf{n}} \\
\mathbf{Z}_{a,\mathbf{n}+L\hat{\mathbf{e}}_b} &= \Gamma_{b,\mathbf{n}}\ \mathbf{Z}_{a,\mathbf{n}}\ \Gamma^\dagger_{b,\mathbf{n}+\hat{\mathbf{e}}_a} \\
\overline{z}_{a,\mathbf{n}+L\hat{\mathbf{e}}_b} &= \Gamma_{b,\mathbf{n}+\hat{\mathbf{e}}_a}\ \overline{z}_{a,\mathbf{n}}\ \Gamma^\dagger_{b,\mathbf{n}} \\
\mathbf{\Xi}_{\mathbf{n}+L\hat{\mathbf{e}}_a} &= \Gamma_{a,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}\ \mathbf{\Xi}_{\mathbf{n}}\ \Gamma^\dagger_{a,\mathbf{n}}\,,
\end{aligned} \tag{4.3}$$

where L is the periodicity of the lattice in two orthogonal directions. The twisted boundary conditions state that after making a cycle on the discrete torus (in either direction) the field matrix at site $\mathbf{n}$ does not necessarily turns back to its original value but to a configuration which is related to it by a gauge transformation. Hence, the fields are not single valued and should be considered as living on the covering space $\mathbb{Z}^2$ of the discrete torus $\mathbb{Z}_L{}^2$. But the covering space $\mathbb{Z}^2$ is in fact bigger then what we need, as will be shown below. The consistency of the field configurations at $\mathbf{n}+L\hat{\mathbf{e}}_a+L\hat{\mathbf{e}}_b$ requires the gauge rotations $\Gamma_{a,\mathbf{n}}$ to satisfy the algebra

$$\Gamma_{a,\mathbf{n}+L\hat{\mathbf{e}}_b}\,\Gamma_{b,\mathbf{n}} = e^{2\pi i Q_{ab}/k}\Gamma_{b,\mathbf{n}+L\hat{\mathbf{e}}_a}\,\Gamma_{a,\mathbf{n}}\,, \tag{4.4}$$

where $Q_{ab}\in\mathbb{Z}$ is the 't Hooft flux and $e^{2\pi i Q_{ab}/k}$ is in the center $\mathbb{Z}_k$ of SU(k) group. In particular, we can choose the gauge rotations to be spacetime independent, with $\Gamma_{a,\mathbf{n}}=\Gamma_a$. In this case, the above algebra turns into

$$\Gamma_a\,\Gamma_b = e^{2\pi i Q_{ab}/k}\Gamma_b\,\Gamma_a\,. \tag{4.5}$$

't Hooft also proved that there are k distinct, non-gauge equivalent choices of boundary conditions, one for each choice of $Q_{ab}\in\mathbb{Z}$ modulo k. The flux matrix Q_{ab} is an antisymmetric matrix and can be written as $Q_{ab}=\epsilon_{ab}\,q$ where $\epsilon_{12}=-\epsilon_{21}=1$. The $e^{2\pi i q/k}$ factor either generates the center $\mathbb{Z}_k$ or a proper subgroup of it. Let the greatest common divisor of q and k be $gcd(q,k)=k_0$. Then we can write $q=k_0q_1$ and $k=k_0k_1$ where q_1 and k_1 are coprime. Thus, we can express the generator

$$e^{2\pi i\frac{q}{k}} = e^{2\pi i\frac{q_1}{k_1}} \qquad \text{with} \qquad gcd(q_1,k_1)=1\,, \tag{4.6}$$

which generates a $\mathbb{Z}_{k_1}$ subgroup of $\mathbb{Z}_k$. A convenient representation for the algebra eq. (4.5) is (in terms of the rank-k matrices) [48, 49]

$$\begin{aligned}
\Gamma_1 &= U^\dagger_{k_1}\otimes\mathbf{1}_{k_0} \\
\Gamma_2 &= V^{q_1}_{k_1}\otimes\mathbf{1}_{k_0}\,.
\end{aligned} \tag{4.7}$$

Now, let us turn back to the covering space. It is easy to see that after making k_1 cycles in either direction on the discrete torus, the field configuration turns back to its original value. Thus, it is sufficient to consider k_1^2-fold cover $\mathbb{Z}^2_{Lk_1}$ of the discrete torus $\mathbb{Z}^2_L$. The fields are periodic on the $Lk_1\times Lk_1$ lattice. The covering space will be essential for constructing the duality.

Next, we will show that the commutative U(k) supersymmetric lattice gauge theory eq. (4.1) on $\mathbb{Z}_L^2$ lattice with twisted boundary conditions eq. (4.3) is equivalent to a noncommutative SYM with reduced rank k_0 on the covering space $\mathbb{Z}_{Lk_1}^2$ with periodic boundary conditions. The action eq. (4.1) includes a sum over all lattice points $\mathbf{n} \in \mathbb{Z}_L^2$ with twisted boundary condition eq. (4.3). As a first step, we can regard the sum over all points on the covering space $\mathbf{n} \in \mathbb{Z}_{Lk_1}^2$ and declare the boundary conditions eq. (4.3) to be a property that the fields living on $\mathbb{Z}_{Lk_1}^2$ must satisfy. This simple observation will lead us to rewrite 't Hooft's twisted boundary conditions as an orbifold with discrete torsion.

We can express the lattice action eq. (4.1) (or eq. (4.2)) with twisted boundary conditions as a matrix theory action on covering space as

$$S = \frac{1}{g^2} \operatorname{Tr}_{(Lk_1)^2} \operatorname{tr}_{(k)} \int d\theta \left[-\frac{1}{2} \widetilde{\mathbf{\Lambda}} \partial_\theta \widetilde{\mathbf{\Lambda}} - \widetilde{\mathbf{\Lambda}} \left[D_a^{-1} \widetilde{\overline{z}}_a , \widetilde{\mathbf{Z}}_a D_a \right] - D_2^{-1} D_1^{-1} \mathbf{\Xi} [\widetilde{\mathbf{Z}}_1 D_1, \widetilde{\mathbf{Z}}_2 D_2] \right], \tag{4.8}$$

where the matrices in the action are rank-$(Lk_1)^2 k$ matrices and the D_a are the commuting lattice shift operators on covering space acting as in eq. (2.26).

In order to write down the twisted boundary conditions in terms of matrix symbols on the covering space, we define the matrix symbol for a gauge rotation matrix as

$$\widetilde{\Gamma}_a = \sum_{\mathbf{n} \in \mathbb{Z}_{Lk_1}^2} \Delta_{\mathbf{n}} \otimes \Gamma_a = \mathbf{1}_{(Lk_1)^2} \otimes \Gamma_a \,. \tag{4.9}$$

Thus, the boundary conditions eq. (4.3) can be captured in terms of a generic matrix symbol $\widetilde{\Phi}$ as

$$(D_a)^L \widetilde{\Phi} (D_a^\dagger)^L = \widetilde{\Gamma}_a \widetilde{\Phi} \widetilde{\Gamma}_a^\dagger \,. \tag{4.10}$$

For example, consider $\mathbf{Z}_1$ which has an r-charge vector $(1,0)$. Using the recipe eq. (2.9), we write it as $\mathbf{Z}_1 = \widetilde{\mathbf{Z}}_1 D_1$ where $\widetilde{\mathbf{Z}}_1$ satisfies eq. (4.10). Thus, $\mathbf{Z}_1$ obeys

$$(D_a)^L \mathbf{Z}_1 (D_a^\dagger)^L = \widetilde{\Gamma}_a \mathbf{Z}_1 (D_1^{-1} \widetilde{\Gamma}_a^\dagger D_1) \tag{4.11}$$

By using eq. (2.26) and eq. (2.24), and taking trace $\operatorname{Tr}_{(Lk_1)^2}$ of both sides, we do recover the twisted boundary conditions given in eq. (4.3). Instead, we can rewrite the constraints as

$$\widetilde{\Phi} = (D_a^\dagger)^L \widetilde{\Gamma}_a \, \widetilde{\Phi} \, \widetilde{\Gamma}_a^\dagger (D_a)^L = \Omega_a^\dagger \, \widetilde{\Phi} \, \Omega_a \qquad \text{with} \qquad \Omega_a = \widetilde{\Gamma}_a^\dagger (D_a)^L \,, \tag{4.12}$$

which is the neutral orbifold constraint given in eq. (2.10), and from which we can obtain eq. (2.1), the starting point of our construction. The orbifold matrices form an $(Lk_1)^2 k$ dimensional projective representation of the cyclic group $\mathbb{Z}_{k_1}$. Thus the orbifold is a $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_1}$ orbifold where orbifold matrices satisfy

$$\Omega_1 \Omega_2 = e^{2\pi i \theta} \Omega_2 \Omega_1 \qquad \text{with} \qquad \theta = \frac{q_1}{k_1} \,. \tag{4.13}$$

The noncommutativity of orbifold matrices is purely because of their 't Hooft Γ_a gauge rotation component. An explicit formula for Ω_a's can be given as

$$\Omega_1 = V_{L'}^L \otimes \mathbf{1}_{L'} \otimes U_{k_1} \otimes \mathbf{1}_{k_0} \,, \qquad \Omega_2 = \mathbf{1}_{L'} \otimes V_{L'}^L \otimes (V_{k_1}^\dagger)^{q_1} \otimes \mathbf{1}_{k_0} \,. \tag{4.14}$$

where $L' = Lk_1$. We have expressed 't Hooft's twisted boundary conditions as an orbifold constraint with discrete torsion on a parent matrix theory of rank $(L')^2k$, and we can use techniques from section 2 to solve it. This interpretation also makes the physical meaning of the discrete torsion clear. Under the given circumstances, the discrete torsion is the ratio of background magnetic flux to the rank of the gauge group. Equivalently, it is the ratio of the topological charges of the theory:

$$\theta = \frac{q}{k} = \frac{q_1}{k_1} = \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \tag{4.15}$$

where ch_0 is the zeroth Chern character (which counts the rank of the gauge group) and ch_1 is first Chern character (which determines the net integer background flux). Hence the noncommutativity parameter is a topological quantity.

The set of constraints eq. (4.10) had been solved rigorously in [13] in a similar context for pure Yang-Mills theory. The main idea is to look for a solution in the k_1^2-fold covering space $\mathbb{Z}^2_{Lk_1}$, k_1 replicas of the lattice in each direction. As we realized, this in turn requires being tolerant to the rank of the deformed parent matrix model. The dual noncommutative SYM gauge theory does not originate from the matrix theory that we used to obtain SYM with magnetic flux Q. The rank of the parent matrix theory which give rise to the ordinary SYM theory with flux is L^2k, whereas the rank of the parent which generates the dual noncommutative theory is $(Lk_1)^2k$. The difference of the ranks of the parent matrix theories comes from the fact that the dual theory lives on the covering space. However, this is not a problem. As we will show later, the total number of elementary degrees of freedom in the commutative theory with flux is equal to the number in the noncommutative theory by a simple counting of momentum modes in both theories. We use the results of section 2 to find the solution to the constraints eq. (4.10) and to obtain the action of the dual theory.

The full set of commutants of the orbifold matrices Ω_a are the basis matrices $Z'_a \otimes \mathbf{1}_{k_0}$ of the noncommutative lattice and are given as

$$Z'_1 = U_{L'} \otimes \mathbf{1}_{L'} \otimes (V_{k_1}^{q_1})^b, \qquad Z'_2 = \mathbf{1}_{L'} \otimes U_{L'} \otimes (U_{k_1}^\dagger)^b \tag{4.16}$$

where b is an integer. The condition for the basis matrices to commute with orbifold matrices requires $\frac{1-q_1b}{k_1} = a$ to be an integer. The equation

$$1 = k_1a + q_1b \tag{4.17}$$

is guaranteed to have a solution for some integers (a, b) by construction since k_1 and q_1 are coprime as in eq. (2.12).

The complete set of solutions to the orbifold conditions is given in terms of momentum basis matrices on the dual torus as

$$\{(J_{\mathbf{p}}^\dagger)' = e^{\pi i\theta' p_1p_2}(Z'_2)^{p_2}(Z'_1)^{p_1} \,|\, \mathbf{p} \in \mathbb{Z}^2_{L'}\} \tag{4.18}$$

with (Z'_a) obeying the commutation relations

$$(Z'_1)(Z'_2) = e^{-2\pi i\theta'}(Z'_2)(Z'_1), \qquad \theta' = \frac{q_1b^2}{k_1} = \frac{b}{k_1} \text{ modulo } \mathbb{Z} \tag{4.19}$$

with noncommutativity parameter θ'. The noncommutativity of the new basis is due to the existence of nonvanishing 't Hooft flux. The parameter θ' is uniquely determined by $\frac{b}{k_1}$ where the integer b is the multiplicative inverse of the magnetic flux q_1 modulo k_1, that is $bq_1 = 1$ modulo k_1. In fact, the solution to the Diophantine equation is unique up to periodicities. Let (a, b) be a solution, then so is $(a + mq_1, b - mk_1)$ where m is any integer. Thus,

$$(a, b) \cong (a + mq_1, b - mk_1) \tag{4.20}$$

defines these congruences.

The most general solution to eq. (4.10) may be written in the new momentum basis as in eq. (2.21)

$$\widetilde{\Phi} = \sum_{\mathbf{p} \in \mathbb{Z}^2_{L'}} (J^\dagger_{\mathbf{p}})' \otimes \Phi_{\mathbf{p}} \tag{4.21}$$

or equivalently in coordinate space basis eq. (2.23) as

$$\widetilde{\Phi} = \sum_{\mathbf{n} \in \mathbb{Z}^2_{L'}} \Delta'_{\mathbf{n}} \otimes \Phi_{\mathbf{n}} \tag{4.22}$$

where $\Delta'_{\mathbf{n}}$ is the dual coordinate basis. The $J'_{\mathbf{p}}$ and $\Delta'_{\mathbf{n}}$ are rank $L'^2 k_1$ matrices which provide the mapping between the $L' \times L'$ lattice and the matrix symbols. The $\Phi_{\mathbf{n}}$ are single valued, rank k_0 fields residing on the dual noncommutative torus.

Substituting the matrix symbols in the new basis into eq. (4.2) and mapping the trace as[7]

$$\mathrm{Tr}_{(L^2)}\ \mathrm{tr}_{(k)} \leftrightarrow \mathrm{Tr}_{(L')^2 k_1}\ \mathrm{tr}_{(k_0)}\ \frac{k}{k_0} \frac{Vol}{Vol'}\,, \tag{4.23}$$

we obtain the following action for the noncommutative $\mathrm{U}(k_0)$ gauge theory:

$$\begin{aligned} S = \frac{k_0}{k}\frac{1}{g^2} \sum_{\mathbf{n} \in \mathbb{Z}^2_{L'}} \mathrm{tr}_{(k_0)} \int d\theta \Big[&-\frac{1}{2}\mathbf{\Lambda}_{\mathbf{n}} \star \partial_\theta \mathbf{\Lambda}_{\mathbf{n}} - \mathbf{\Lambda}_{\mathbf{n}} \star (\overline{z}_{a,\mathbf{n}-\hat{\mathbf{e}}_a} \star \mathbf{Z}_{a,\mathbf{n}-\hat{\mathbf{e}}_a} - \mathbf{Z}_{a,\mathbf{n}} \star \overline{z}_{a,\mathbf{n}}) - \\ &- \mathbf{\Xi}_{\mathbf{n}} \star (\mathbf{Z}_{1,\mathbf{n}} \star \mathbf{Z}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - \mathbf{Z}_{2,\mathbf{n}} \star \mathbf{Z}_{1,\mathbf{n}+\hat{\mathbf{e}}_2}) \Big]\,. \end{aligned} \tag{4.24}$$

We have shown that at a nonperturbative level a $\mathrm{U}(k)$ SYM theory with a background 't Hooft flux (hence multivalued fields) is equivalent to a purely noncommutative gauge theory $\mathrm{U}(k_0)$ with periodic boundary conditions (hence single valued fields). The rank k_0 of the noncommutative theory is less than or equal to the rank of the commutative one. The parameters of the theory on the commutative lattice are the number of lattice sites $L \times L$, the coupling constant g, the background magnetic flux $q = q_1 k_0$, and the rank of the gauge group $k = k_0 k_1$ where k_0 is greatest common divisor of k and q. The parameters in its noncommutative equivalent are the number of lattice sites $L' \times L'$, the noncommutativity parameter θ', the reduced rank of the gauge group k_0, and the coupling constant g'. The relations among the parameters can be ex-

[7] The traces are normalized such that Tr **1** gives the volume of discrete torus and tr **1** is the rank of the gauge group. The other factors are there for the relative normalization. Namely, we request that the double trace of the corresponding identity matrices on both sides to be equal to each other.

pressed as

$$
\begin{aligned}
(L')^2 &= L^2 k_1^2 = L^2 \frac{k^2}{k_0^2} \\
\theta' &= \frac{b}{k_1}\,, \qquad bq_1 = 1 \text{ modulo } k_1 \\
g' &= g\left(\frac{k}{k_0}\right)^{1/2}
\end{aligned}
\tag{4.25}
$$

where b is an integer. Notice that the total number of elementary degrees of freedom are same in both theories. Even though the ranks of the gauge groups are different, the volume of the dual torus is scaled in such a way that the total number of degrees of freedom is kept fixed. In the commutative case, we have L^2 unit cells (momentum modes) on the real lattice (momentum lattice) and each cell (mode) has the matter content of the $\mathcal{N} = (2,2)$ multiplet of $d = 2$, U(k) supersymmetric Yang-Mills theory. Hence there are L^2k^2 degrees of freedom modulo matter content of $\mathcal{N} = (2,2)$ which is irrelevant for our discussion. In the noncommutative case, the lattice has $(L')^2 = L^2k_1^2$ cells (momentum modes) and the gauge group is U(k_0) giving another factor of k_0^2. Thus, a total of $L^2k_1^2k_0^2$ degrees of freedom which is equal to the one in the commutative lattice.

There are various cases that are particularly interesting: If q and k are coprime, the flux q sector of the U(k) theory is equivalent to a noncommutative U(1) theory. Notice that the U(1) in U(k) is free, noninteracting and hence decouples from the U(k), leaving the SU(k) sector. The nonabelian features of the ordinary SU(k) theory are captured by the spacetime noncommutativity of the U(1) theory.

Let us assume k to be a prime number. In that case, each of the $k-1$ nontrivial topological sectors is mapped to a U(1) theory with quantized units of noncommutativity θ'. For the sector with flux q, the noncommutativity is $b\frac{1}{k}$ with $b.q = 1$ modulo k. Thus θ' in dual U(1) theories is quantized in units of $\frac{1}{k}$.

Nontrivial topological sectors of noncommutative gauge theory. Let us consider a U(k) supersymmetric gauge theory with a finite noncommutativity parameter $\tilde{\theta}$ in the background magnetic flux on an $L \times L$ discrete torus. Thus the fields on the noncommutative lattice are multivalued, and we would like to rewrite this theory as a purely noncommutative theory on some discrete torus with size $L' \times L'$ with periodic boundary conditions. Following similar considerations as above, we find that the dual theory is related by an SL$(2,\mathbb{Z})$ transformation to the theory we started with. The SL$(2,\mathbb{Z})$ acts as

$$
\begin{aligned}
(L')^2 &= L^2(k_1 + q_1\tilde{\theta})^2 \\
\theta' &= \frac{b - a\tilde{\theta}}{k_1 + q_1\tilde{\theta}} \qquad \text{modulo } \mathbb{Z} \\
g' &= g(k_1 + q_1\tilde{\theta})^{1/2}
\end{aligned}
\tag{4.26}
$$

where the matrix

$$\begin{pmatrix} b & -a \\ k_1 & q_1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{4.27}$$

Notice that for $\tilde{\theta} = 0$, the transformations eq. (4.26) reproduce eq. (4.25). From eq. (2.20), we know that $\tilde{\theta}$ is not arbitrary, it satisfies $L\tilde{\theta} \in 2\mathbb{Z}$. Thus the size of the dual torus is $L' = Lk_1 + q_1 L\tilde{\theta}$ which is an integer as it should be. This result appeared in string and M-theory as part of the T duality group [1, 4, 52 − 55] and in the context of fully regularized pure Yang-Mills in [15]. In terms of dual gravity picture, it is discussed in [56, 57]. Hence, we will not discuss it here in detail.

However, let us reconsider the U(k) gauge theory with a prime k and its nontrivial topological sectors. These sectors have purely noncommutative U(1) duals. If the q flux sector is equivalent to a theory with noncommutativity $\tilde{\theta}$, then the $q' = k - q$ flux sector is equivalent to a theory with noncommutativity θ' such that $\tilde{\theta} + \theta' = 1$. The U(1) theory with noncommutativity θ' is dual to a U(1) with noncommutativity $\tilde{\theta}$ by an SL$(2,\mathbb{Z})$ transformation, which is given by $S = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

5. Noncommutative $\mathcal{N} = (4,4)$ in $d = 2$

The prescription we described in section 3 can be applied almost verbatim to generate a regularization for noncommutative $\mathcal{N} = (4,4)$ extended SYM theories in $d = 2$ dimension.

The action of the parent matrix theory is a $\mathcal{Q} = 2$ the deformation of dimensional reduction of $\mathcal{N} = 1$ SYM theory with U(N_c) gauge group from $d = 6$ Euclidean dimensions to zero dimension. The undeformed zero dimensional model possess a $\mathcal{Q} = 8$ supersymmetry and a global SO(6) × SU(2) R-symmetry group. The SO(6) is inherited from the Lorentz symmetry and SU(2) is the R-symmetry group prior to reduction. The matter content of the theory is a six component gauge vector potential $\{z_a, \bar{z}_a\}$, $a = 1, \ldots 3$ expressed as complex matrices and an eight component spinor $\{\xi_a, \psi_a, \chi, \lambda\}$, $a = 1, \ldots 3$. There are also auxiliary fields $\{d, G, \overline{G}\}$ introduced for off-shell supersymmetry.

We begin with a little digression to $\mathcal{Q} = 2$ supersymmetry and its supermultiplets in terms of which the deformed matrix model action can be written in manifestly $\mathcal{Q} = 2$ form [21]. We will work with a superspace with two independent Grassmann valued coordinates $(\theta, \bar{\theta})$. There are two important nilpotent symmetries, Q and $\overline{Q}$. The action of supersymmetry Q on component fields is

$$\begin{aligned}
Qz_i &= \sqrt{2}\,\psi_i & Q\bar{z}_i &= 0 \\
Q\psi_i &= 0 & Q\xi_i &= -2\,[\bar{z}_3,\, \bar{z}_i] \\
Qz_3 &= \sqrt{2}\psi_3 & Q\lambda &= -\,(id + [\bar{z}_3,\, z_3]) \\
Q\psi_3 &= 0 & Q(id) &= -\sqrt{2}[\bar{z}_3,\, \psi_3] \\
Q\xi_3 &= \sqrt{2}G & Q\chi &= 0 \\
QG &= 0 & Q\overline{G} &= -2[\bar{z}_3, \chi]\,,
\end{aligned} \tag{5.1}$$

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and $Q\bar{z}_3 = 0$ where $i = 1, 2$. The other supersymmetry $\overline{Q}$ acts as

$$\begin{array}{llll}
\overline{Q} z_i &= 0 & \overline{Q} \bar{z}_i &= \sqrt{2}\, \xi_i \\
\overline{Q} \psi_i &= -2\, [\bar{z}_3, z_i] & \overline{Q} \xi_i &= 0 \\
\overline{Q} z_3 &= \sqrt{2} \lambda & \overline{Q} \psi_3 &= (id - [\bar{z}_3,\, z_3]) \\
\overline{Q} \lambda &= 0 & \overline{Q} (id) &= \sqrt{2} [\bar{z}_3, \lambda] \\
\overline{Q} \xi_3 &= 0 & \overline{Q} \chi &= \sqrt{2}\, \overline{G} \\
\overline{Q} G &= -2 [\bar{z}_3, \xi_3] & \overline{Q}\, \overline{G} &= 0\,,
\end{array} \tag{5.2}$$

and $\overline{Q}\bar{z}_3 = 0$ where $i = 1, 2$. Since $\bar{z}_3$ is in the intersection of the kernel of Q and $\overline{Q}$, it is a supersymmetry singlet. In order to construct the supersymmetry multiplets, we introduce two differential operators Q and $\overline{Q}$, realized in the superspace as

$$Q = \frac{\partial}{\partial \theta} + \sqrt{2}\, \bar{\theta} [\bar{z}_3, \,\cdot\,]\,, \qquad \overline{Q} = \frac{\partial}{\partial \bar{\theta}} + \sqrt{2}\, \theta [\bar{z}_3, \,\cdot\,]\,. \tag{5.3}$$

with the algebra

$$Q^2 = \overline{Q}^2 = 0\,, \qquad \{Q, \overline{Q}\} \cdot = 2\sqrt{2} [\bar{z}_3, \,\cdot] \tag{5.4}$$

Notice that the supersymmetry transformations eq. (5.1) and eq. (5.2) are satisfied off-shell.[8] We can also define supersymmetric derivatives which anticommute with the Q's:

$$\mathcal{D} = \frac{\partial}{\partial \theta} - \sqrt{2}\, \bar{\theta}\, [\bar{z}_3, \,\cdot\,]\,, \qquad \overline{\mathcal{D}} = \frac{\partial}{\partial \bar{\theta}} - \sqrt{2}\, \theta\, [\bar{z}_3, \,\cdot\,]\,. \tag{5.5}$$

The supersymmetric derivatives may be used in constructing the action.

The transformations in the first two lines of eq. (5.1) and eq. (5.2) are compatible with bosonic chiral and antichiral superfields with a superspace expansion

$$\begin{aligned}
\mathbf{Z}_i &= z_i + \sqrt{2}\, \theta \psi_i - \sqrt{2}\, \theta \bar{\theta} [\bar{z}_3, z_i]\,, \\
\overline{\mathbf{Z}}_i &= \bar{z}_i + \sqrt{2}\, \bar{\theta}\, \xi_i + \sqrt{2}\, \theta \bar{\theta} [\bar{z}_3, \bar{z}_i]\,,
\end{aligned} \tag{5.6}$$

satisfying the chiral and antichiral constraints $\overline{\mathcal{D}}\mathbf{Z}_i = 0$, and $\mathcal{D}\overline{\mathbf{Z}}_i = 0$. The last two lines of the transformations eq. (5.1) and eq. (5.2) are in accord with fermi multiplets with superspace expansions

$$\begin{aligned}
\mathbf{\Xi} &= \xi_3 + \sqrt{2}\, \theta G - \sqrt{2}\, \theta \bar{\theta} [\bar{z}_3, \xi_3]\,, \\
\overline{\mathbf{\Xi}} &= \chi + \sqrt{2}\, \bar{\theta} \overline{G} + \sqrt{2}\ \theta \bar{\theta} [\bar{z}_3, \chi]\,,
\end{aligned} \tag{5.7}$$

satisfying chiral and antichiral constraints $\overline{\mathcal{D}}\mathbf{\Xi} = 0$ and $\mathcal{D}\overline{\mathbf{\Xi}} = 0$. The transformations in the middle two lines of eq. (5.1) and eq. (5.2) combine to a vector multiplet [9]

$$\mathbf{S} = z_3 + \sqrt{2}\, \theta \psi_3 + \sqrt{2}\, \bar{\theta} \lambda + \sqrt{2}\, \theta \bar{\theta} (id)\,. \tag{5.8}$$

[8] The supersymmetry algebra is the same as eq. (5.4), the algebra of differential operators, except that the sign of the right hand side of the anticommutator $\{Q, \overline{Q}\}$ is switched. For an explanation, see [58], page 25-26.

[9] The vector multiplet in Minkowski space satisfies a reality (Hermiticity) condition. The Euclidean space vector multiplet should be interpreted as satisfying the reality condition after analytical continuation to Minkowski space. The complex matrices z_3 and $\bar{z}_3$ turn into hermitean matrices upon analytic continuation. (For example, compare with the dimensional reduction of $(0, 2)$ multiplets appearing in [51] to zero dimension) In this regard, λ and ψ_3 which are independent Grassmann variables in euclidean space turn into complex conjugate variables in Minkowski space. (similarly for fermionic components of bosonic chiral and antichiral multiplets and highest components of fermi multiplets.)

	$\mathbf{Z}_1$	$\overline{\mathbf{Z}}_1$	$\mathbf{Z}_2$	$\overline{\mathbf{Z}}_2$	$\mathbf{S}, \mathbf{\Upsilon}, \overline{\mathbf{\Upsilon}}$	$\mathbf{\Xi}$	$\overline{\mathbf{\Xi}}$
r_1	$+1$	-1	0	0	0	-1	$+1$
r_2	0	0	$+1$	-1	0	-1	$+1$

Table 2: The $r_{1,2}$ charges of the fields of the $\mathcal{Q}=2$ deformed matrix theory which define the orbifold projection.

From the vector superfield we can create the chiral and anti-chiral superfields which will appear in the action:

$$\begin{aligned}\mathbf{\Upsilon} &= \frac{\overline{\mathcal{D}}\mathbf{S}}{\sqrt{2}} = \lambda - \theta(+[\bar{z}_3, z_3] + id) - \sqrt{2}\theta\bar{\theta}[\bar{z}_3, \lambda]\,, \\ \overline{\mathbf{\Upsilon}} &= \frac{\mathcal{D}\mathbf{S}}{\sqrt{2}} = \psi_3 + \bar{\theta}(-[\bar{z}_3, z_3] + id) + \sqrt{2}\theta\bar{\theta}[\bar{z}_3, \psi_3]\,. \end{aligned} \tag{5.9}$$

with $\overline{\mathcal{D}}\mathbf{\Upsilon} = 0$ and $\mathcal{D}\overline{\mathbf{\Upsilon}} = 0$.

Note that the θ ($\bar{\theta}$) components of (anti-)chiral superfields, and the $\bar{\theta}\theta$ components of a general superfield, transform under supersymmetry into a commutator; therefore, the trace of such terms are supersymmetric invariants and are suitable for construction of the action. In terms of these superfields, the action of the $\mathcal{Q}=2$ deformed matrix model may be written as

$$\begin{aligned} S = \frac{1}{g^2}\int d\theta d\bar{\theta}\ \mathrm{Tr}\left(\frac{1}{2}\overline{\mathbf{\Upsilon}}\mathbf{\Upsilon} + \frac{1}{\sqrt{2}}\overline{\mathbf{Z}}_i[\mathbf{S}, \mathbf{Z}_i] + \frac{1}{2}\overline{\mathbf{\Xi}}\mathbf{\Xi}\right) + \\ + \int d\theta\ \mathrm{Tr}\left(\mathbf{\Xi}\,[\mathbf{Z}_1, \mathbf{Z}_2]_\zeta\right) - \int d\bar{\theta}\ \mathrm{Tr}\left(\overline{\mathbf{\Xi}}\,[\overline{\mathbf{Z}}_2, \overline{\mathbf{Z}}_1]_{\zeta^*}\right), \end{aligned} \tag{5.10}$$

where ζ can be regarded as a deformation of the superpotential and is set to a specific value following the discussion in 3.2. Similar to section 3, we can orbifold the $\mathcal{Q}=2$ deformed matrix model to obtain the lattice action for $\mathcal{N}=(4,4)$ noncommutative SYM theory where the lattice possess $\mathcal{Q}=2$ exact supersymmetries. To orbifold, we use conveniently chosen **r**-charges, given in Table 2.

The lattice action for noncommutative $\mathcal{N}=(4,4)$ SYM theory with gauge group $\mathrm{U}(k)$ is written with manifestly $\mathcal{Q}=2$ superfields as

$$\begin{aligned} S = \frac{1}{g_{nc}^2}\sum_{\mathbf{n}\in\mathbb{Z}_L^2} \mathrm{tr}_{(k)}\Bigg[\int d\theta d\bar{\theta}\Bigg(\frac{1}{2}\overline{\mathbf{\Upsilon}}_{\mathbf{n}}\star\mathbf{\Upsilon}_{\mathbf{n}} + \frac{1}{\sqrt{2}}\mathbf{S}_{\mathbf{n}}\star \\ \star\left(\mathbf{Z}_{a,\mathbf{n}}\star\overline{\mathbf{Z}}_{a,\mathbf{n}} - \overline{\mathbf{Z}}_{a,\mathbf{n}-\hat{\mathbf{e}}_a}\star\mathbf{Z}_{a,\mathbf{n}-\hat{\mathbf{e}}_a}\right) + \frac{1}{2}\overline{\mathbf{\Xi}}_{\mathbf{n}}\star\mathbf{\Xi}_{\mathbf{n}}\Bigg) + \\ + \int d\theta\left(\mathbf{\Xi}_{\mathbf{n}}\star\left(\mathbf{Z}_{1,\mathbf{n}}\star\mathbf{Z}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - \mathbf{Z}_{2,\mathbf{n}}\star\mathbf{Z}_{1,\mathbf{n}+\hat{\mathbf{e}}_2}\right)\right) + \mathrm{a.h.}\Bigg], \end{aligned} \tag{5.11}$$

where *a.h.* stands for the anti-holomorphic part of the superpotential. The chiral and antichiral superfields forming the lattice action satisfy highly nonlocal constraints, such as

$$\overline{\mathcal{D}}\mathbf{Z}_{1,\mathbf{n}} = \frac{\partial}{\partial\theta}\mathbf{Z}_{1,\mathbf{n}} - \sqrt{2}\,\bar{\theta}\left(\bar{z}_{3,\mathbf{n}}\star\mathbf{Z}_{1,\mathbf{n}} - \mathbf{Z}_{1,\mathbf{n}}\star\bar{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1}\right) = 0 \tag{5.12}$$

	$\mathbf{Z}_1$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{V}$
r_1	$+1$	0	-1	0
r_2	0	$+1$	-1	0

Table 3: The $r_{1,2}$ charges of the fields of the $\mathcal{Q} = 4$ deformed matrix theory which define the orbifold projection. The charges of the antichiral multiplets are negative of their chiral counterparts.

which implies some components of the superfields are nonlocal. The explicit form of the lattice superfields can be easily obtained by using eq. (2.21). The discussions of the continuum limits, the reason for deformation of the superpotential and discussion of dualities are the same as $\mathcal{N} = (2, 2)$ SYM theory and will not be repeated here.

6. Noncommutative $\mathcal{N} = (8, 8)$ in $d = 2$

The last target theory for which we construct a lattice regularization is maximally supersymmetric $\mathcal{N} = (8, 8)$ noncommutative SYM theory in two dimensions. The action of the deformed matrix theory is a $\mathcal{Q} = 4$ deformation of the dimensional reduction of $\mathcal{N} = 1$ SYM theory from $d = 10$ dimensions to $d = 0$ dimension. The $\mathcal{Q} = 4$ supersymmetry is the $\mathcal{N} = 1$ supersymmetry of the $d = 4$ dimensions, hence the multiplets are familiar from supersymmetry in four dimensions (see for example [58]). We will denote the three chiral (antichiral) multiplets with $\mathbf{Z}_a$ ($\overline{\mathbf{Z}}_a$), the vector multiplet with $\mathbf{V}$ and the chiral (antichiral) fermi multiplet with $\mathbf{W}, \overline{\mathbf{W}}$. The holomorphic superpotential of the $\mathcal{Q} = 4$ deformed matrix model is

$$\mathrm{Tr}\,(\mathbf{Z}_3[\mathbf{Z}_1, \mathbf{Z}_2]_\zeta)\,, \tag{6.1}$$

where ζ is the deformation parameter.[10] For $\zeta = 1$, the matrix model possess all $\mathcal{Q} = 16$ supersymmetries and a global SO(10) R-symmetry.

The orbifold projection of the $\mathcal{Q} = 4$ deformed matrix model can be performed by using the **r**-charges under the $\mathrm{U}(1) \times \mathrm{U}(1)$ subgroup of the global symmetry group of the matrix model given in table 3.

The lattice action preserves four exact supersymmetries. For each unit cell, there are three chiral multiplets and three antichiral ones residing on the links. The vector multiplet resides on the sites. The lattice action for noncommutative $\mathcal{N} = (8, 8)$ theory may be written in manifestly $\mathcal{Q} = 4$ supersymmetric form as

$$\begin{aligned} S = \frac{1}{g_{nc}^2} \sum_{\mathbf{n} \in \mathbb{Z}_L^2} \mathrm{Tr} \bigg[& \int d^2\theta d^2\bar{\theta} \overline{\mathbf{Z}}_{a,\mathbf{n}} \star e^{2\mathbf{V}_\mathbf{n}} \star \mathbf{Z}_{a,\mathbf{n}} \star e^{-2\mathbf{V}_{\mathbf{n}+\hat{\mathbf{e}}_a}} + \frac{1}{4} \int d^2\theta\, \mathbf{W}_\mathbf{n}^\alpha \star \mathbf{W}_{\alpha,\mathbf{n}} + \text{a.h.} + \\ & + \sqrt{2} \int d^2\theta\, \mathbf{Z}_{3,\mathbf{n}} \star (\mathbf{Z}_{1,\mathbf{n}} \star \mathbf{Z}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - \mathbf{Z}_{2,\mathbf{n}} \star \mathbf{Z}_{1,\mathbf{n}+\hat{\mathbf{e}}_2}) + \text{a.h.} \bigg]\,, \end{aligned} \tag{6.2}$$

where a.h. stands for the antiholomorphic superpotential. The first term in the action is the standard Kähler term, and it also reflects the star gauge transformation properties of the

[10] In the four dimensional counterpart of this matrix model, the deformation of the superpotential eq. (6.1) is an exactly marginal deformation and it was examined in [59].

chiral link fields. The second term must be interpreted as an holomorphic superpotential of a chiral fermi multiplet $\mathbf{W}_\alpha$ but not as a field strength superfield (similarly for its antiholomorphic counterpart). The last term, which is a superpotential gives rise to gauge kinetic terms (as well as other terms) in the continuum target theory. We would like to emphasize that the gauge boson of the continuum theory does not reside in the site multiplet $\mathbf{V}$, but on the chiral link multiplets. Similarly, the scalars of the continuum theory (there are eight of them) live both on the chiral link multiplets and on the site multiplet $\mathbf{V}$. For more details of the commutative counterpart of this lattice, see [22, section 5].

7. Discussion and outlook

In this article, we have examined nonperturbative aspects of extended SYM theories in $d = 2$ Euclidean dimensions. We first constructed a full star gauge invariant and manifestly supersymmetric non-perturbative regulator for noncommutative supersymmetric gauge theories.

Then we observed that SYM theories may be classified as (i) noncommutative, with no background flux, (ii) noncommutative, with a background flux, (iii) commutative, with a background flux, (iv) commutative, with no background flux. The theories in the first three class are related to each other by an $\mathrm{SL}(2,\mathbb{Z})$ transformation. In particular, we have explicitly shown that the nontrivial topological sectors of SYM theories may be rewritten as purely noncommutative field theories. The commutative sector with no background flux is not related by an $\mathrm{SL}(2,\mathbb{Z})$ to other sectors.

The noncommutative SYM theory with rank k_0 and noncommutativity parameter θ' given in eq. (4.24) may be regarded as regulated via a commutative field theory eq. (4.1) with rank k and background flux q obeying twisted boundary conditions eq. (4.3). This brings us to the question of renormalizability and the continuum of quantum theory. In [20, 21], it has been shown that the target continuum theory of the commutative lattice action is obtained without any fine tuning owing to the superrenormalizability of the theory (modulo an uninteresting Fayet-Illiopoulos term). Also, it has been argued that the fluctuation of the modulus in the quantum theory does not destroy the lattice interpretation as long as the continuum limit is taken such that $\mathfrak{a}^2 g_2^2 \ln(L\mathfrak{a}/\mathfrak{a}) \to 0$. (For a fuller explanation, see [20, 21].) We believe that these two facts imply the corresponding quantum noncommutative theory is a renormalizable theory with a sensible continuum limit.

The observables in the zero momentum sector of the commutative gauge theory with flux can be mapped to the star gauge invariant observables which also carry zero momentum in the noncommutative theory. In the case of pure Yang-Mills theory, the mapping of the zero momentum sector along with the translational invariance of the two theories may be used to obtain the loop equations and to show the relation among at least a subset of observables in two theories along the lines of [60] in the large N_c limit.

On the other hand, the noncommutative SYM theories we have examined have no local gauge invariant observables. The noncommutativity of spacetime puts a stringent bound on the size of a gauge invariant excitation. In particular, if we want to minimize smearing of an excitation in both directions, the bound $\Delta x_1 \Delta x_2 \gtrsim \Theta_{12}$ requires that $\Delta x_a \sim \sqrt{\Theta}$

assuming Θ is larger then the other length scale in the problem $1/g_2^2$. Thus the best localization that can be achieved in the theory is (in terms of lattice parameters) $L\mathfrak{a}\sqrt{\theta'}$ where L^2 is the number of sites and $\mathfrak{a}$ is lattice spacing and $\theta' \in [0, 1)$ is dimensionless noncommutativity on the lattice. We see that the dimensionless ratio of the size of an excitation to the size of the box is

$$\frac{\Delta x_a}{L\mathfrak{a}} \sim \sqrt{\theta'} \,. \tag{7.1}$$

The existence of the Seiberg-Witten map [4] from commutative gauge fields to noncommutative gauge fields suggests that the local gauge invariant observables in commutative theory may be used to construct the nonlocal gauge invariant counterparts in the noncommutative theory where the latter are smeared over a region of size $\sqrt{\Theta}$ [10] consistent with what we argued above. For observables in noncommutative field theories, see [61].

One other issue we did not discuss is the irrational dimensionless noncommutativity parameters, which requires taking the infinite N_c limit [15] of ordinary lattice SYM theory. The large but finite N_c gauge theories have flux sectors which are equivalent to a finite rank gauge theories with rational noncommutativity parameters.

The study of three and four dimensional noncommutative extended SYM theories is left for the future work. Even though generalization of the results of this paper is straightforward, the theories in three and four dimensions are richer both topologically and dynamically. For example, in four dimensions, the theory has sectors with both electric and magnetic fluxes as well as instantons. Following [40] and the arguments given in section 4, the dualities between different noncommutative theories can be established rigorously at the non-perturbative level.

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